

Slant products on the Higson–Roe exact sequence

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Abstract

We construct a slant product $/: S_p(X \times Y) \times K_{1-q}(\mathfrak{c}^{\text{red}}Y) \rightarrow S_{p-q}(X)$ on the analytic structure group of Higson and Roe and the K-theory of the stable Higson corona of Emerson and Meyer. The latter is the domain of the co-assembly map $\mu^*: K_{1-*}(\mathfrak{c}^{\text{red}}Y) \rightarrow K^*(Y)$. In fact, we obtain such products simultaneously on the entire Higson–Roe sequence. The existence of these products implies injectivity results for external product maps on each term of the Higson–Roe sequence. Our results apply in particular to taking products with aspherical manifolds whose fundamental groups admit coarse embeddings into Hilbert space. To conceptualize the general class of manifolds where this method applies, we introduce the notion of Higson-essentialness. A complete spin^c -manifold is Higson-essential if its fundamental class is detected by the stable Higson corona via the co-assembly map. We prove that coarsely hypereuclidean manifolds are Higson-essential. Finally, we draw conclusions for positive scalar curvature metrics on product spaces, in particular on non-compact manifolds. We also construct suitable equivariant versions of these slant products and discuss related problems of exactness and amenability of the stable Higson corona.

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1 Introduction

The first main entity we study in this paper is the analytic exact sequence of Higson and Roe [HR05]. Questions around this sequence have generated substantial activity in higher index theory, for a selection of recent works see [PS14; XY14; PS16; Zen17; DG17; PZ19; WXY16]. Its original motivation as devised by Higson and Roe was to serve as the target of certain analytic index maps defined on the surgery sequence from geometric topology. For this reason it also often called “analytic surgery sequence”. Later it was also used to serve as a target for index maps on the positive scalar curvature sequence of Stolz [PS14; XY14] (see Section 1.3).

To describe the sequence, start with the following setup. Let X be a proper metric space endowed with a proper action of a countable discrete group G . Then

the *analytic sequence of Higson and Roe* associated to X reads as follows

$$\cdots \rightarrow K_{*+1}(C_G^*X) \xrightarrow{\partial} S_*^G(X) \rightarrow K_*^G(X) \xrightarrow{\text{Ind}} K_*(C_G^*X) \rightarrow \cdots, \quad (1.1)$$

where $K_*(C_G^*X)$ is the topological K-theory of the *equivariant Roe algebra* C_G^*X , $S_*^G(X)$ is the *analytic structure group* of X and $K_*^G(X)$ stands for the *equivariant locally finite K-homology* of X . The definitions of these groups are given in Section 3. If X is a complete spin^c -manifold of dimension m , where the spin^c -structure is preserved by G , then the Dirac operator defines a fundamental class $[\not{D}_X] \in K_m^G(X)$.

If the G -action is free, then $K_*^G(X) \cong K_*(G \backslash X)$. If the G -action on X is cocompact, then C_G^*X is canonically Morita equivalent to the reduced group C^* -algebra C_{red}^*G . The sequence (1.1) is often applied to the case $X = \widetilde{M}$, where \widetilde{M} is the universal covering of a compact smooth manifold or finite simplicial complex M , and $G = \pi_1 M$. In this case it becomes

$$\cdots \rightarrow K_{*+1}(C_{\text{red}}^*G) \xrightarrow{\partial} S_*^G(\widetilde{M}) \rightarrow K_*(M) \xrightarrow{\text{Ind}_G} K_*(C_{\text{red}}^*G) \rightarrow \cdots. \quad (1.2)$$

Our initial investigations started with the following result of Zeidler:

Theorem 1.1 ([Zei16a, Corollary 5.8]). *Let N be a closed spin^c -manifold¹ of dimension n such that its universal covering \widetilde{N} is (rationally) stably hypereuclidean. Then for every closed manifold M the map*

$$S_*^G(\widetilde{M}) \rightarrow S_{*+n}^{G \times H}(\widetilde{M} \times \widetilde{N}), \quad x \mapsto x \times [\not{D}_N],$$

where $G = \pi_1 M$ and $H = \pi_1 N$, is (rationally) split-injective.

Here we implicitly used an external product which mixes an element of the structure group and an element of K-homology to produce an element of the structure group of the product. This is a well-known product construction which we revisit in Section 3.

Next, recall the notion of a (stably) hypereuclidean manifold. This goes back to Gromov and Lawson [GL83].

Definition 1.2. A complete oriented Riemannian manifold X of dimension m is called *stably hypereuclidean* if for some $k \in \mathbb{N}$, the product $X \times \mathbb{R}^k$ admits a proper Lipschitz map to Euclidean space \mathbb{R}^{m+k} of degree 1. If the latter condition is relaxed to merely non-zero degree, it is called *rationally stably hypereuclidean*. \blacklozenge

The assumption of being (rationally) stably hypereuclidean is quite general. Dranishnikov [Dra06] proved that the universal cover of any closed aspherical manifold whose fundamental group has finite asymptotic dimension is stably hypereuclidean. Further, there is currently no known example of an aspherical closed manifold whose universal covering is not (rationally) stably hypereuclidean.

This notion is closely related to the strong Novikov conjecture. Indeed, if the universal covering \widetilde{M} of a closed spin^c -manifold M is rationally hypereuclidean, then the higher index class $\text{Ind}_G([\not{D}_M]) \in K_*(C_{\text{red}}^*G)$ is non-zero. In fact, since \mathbb{R}^n satisfies the coarse Baum–Connes conjecture, the *coarse index* $\text{Ind}([\not{D}_{\widetilde{M}}]) \in K_*(C^*\widetilde{M})$ does not vanish. The canonical forgetful map $K_*(C_{\text{red}}^*G) = K_*(C_G^*\widetilde{M}) \rightarrow K_*(C^*(\widetilde{M}))$ takes the higher index to the coarse index and so this also proves non-vanishing of former. Furthermore, it turns out that the conclusion of Theorem 1.1 in itself implies non-vanishing of the higher index class:

¹Note that, although Corollary 5.8 of [Zei16a] is stated for spin -manifolds, the statement makes perfectly sense for spin^c -manifolds and the proof given by Zeidler works also perfectly well in the more general case of spin^c -manifolds.

Theorem 1.3 (see Theorem 2.1). *Let $z \in K_n(N)$ be such that for every closed manifold M the map $S_*^G(\widetilde{M}) \rightarrow S_{*+n}^{G \times H}(\widetilde{M} \times \widetilde{N})$, $x \mapsto x \times z$, where $G = \pi_1 M$ and $H = \pi_1 N$, is rationally injective. Then $\text{Ind}_H(z) \in K_n(C_{\text{red}}^*H)$ is rationally non-zero.*

The analogous integral version of Theorem 2.1 also holds. In fact, it is formally weaker because injectivity implies rational injectivity and rational non-vanishing implies non-vanishing.

These considerations suggested to us that there ought to be analytic conditions related to the coarse Baum–Connes conjecture and the strong Novikov conjecture that are weaker than hypereuclidean and would imply the conclusion Theorem 1.1. The quest for such hypotheses lead us to to the second main player of the present paper—the *coarse co-assembly map*

$$\mu^* : K_{1-*}(\mathfrak{c}^{\text{red}}X) \rightarrow K^*(X)$$

of Emerson and Meyer [EM06], where $\mathfrak{c}^{\text{red}}X$ denotes the stable Higson corona of X and $K^*(X)$ is the compactly supported K-theory of X . The coarse co-assembly map is dual to the coarse index map (or “coarse assembly map”)

$$\mu = \text{Ind} : K_*(X) \rightarrow K_*(C^*X)$$

via a pairing between the K-theories of the stable Higson corona and the Roe algebra. In particular, a K-homology class which pairs non-trivially with a K-theory class in the image of μ^* does not lie in the kernel of Ind . Our first main result extends Theorem 1.1 to such classes. To formulate this, we introduce the following new property.

Definition 1.4. We say that a complete Riemannian spin^c -manifold X of dimension m is (rationally) *Higson-essential* if there exists $\vartheta \in K_{1-m}(\mathfrak{c}^{\text{red}}X)$ such that $\langle [\not{D}_X], \mu^*(\vartheta) \rangle = 1$ ($\neq 0$, respectively), where $[\not{D}_X] \in K_m(X)$ denotes the K-homological fundamental class of the spin^c -structure. \blacklozenge

This notion is reminiscent of coarse largeness properties à la Brunnbauer and Hanke [BH10]. In fact, we will see that (rationally) *coarsely stably hypereuclidean* spin^c -manifolds are (rationally) Higson-essential (see Theorem 6.11). In particular, this applies to (rationally) stably hypereuclidean manifolds.

Furthermore, if the co-assembly map $\mu^* : K_{1-*}(\mathfrak{c}^{\text{red}}X) \rightarrow K^*(X)$ is surjective, then a spin^c -manifold X is automatically Higson-essential. If $X = \widetilde{M}$ is the universal covering of a closed *aspherical* manifold, then this co-assembly map is in fact known to be an isomorphism for a very broad class of examples. For instance, if $\pi_1 M$ is coarsely embeddable into a Hilbert space [EM06, Theorem 9.2] or more generally if $\pi_1 M$ has a γ -element [EM07, Corollary 34], or if it admits an expanding and coherent combing [EW17, Theorem 5.10].

Our first main result generalizes Theorem 1.1 from stably hypereuclidean to Higson-essential. Start with a more general version which applies to non-compact manifolds.

Theorem 1.5 (see Corollary 6.12). *Let Y be an n -dimensional complete spin^c -manifold of continuously bounded geometry². Suppose that Y is (rationally) Higson-essential. Assume furthermore that Y is endowed with a proper action of a countable discrete group H which preserves the spin^c -structure.*

²see Definition 4.1.(b)

Then for every proper metric space X which is endowed with a proper action of a countable discrete group G , the external product maps

$$S_*^G(X) \rightarrow S_{*+n}^{G \times H}(X \times Y), \quad x \mapsto x \times [\mathcal{D}_Y]$$

and

$$K_*(C_G^* X) \rightarrow K_{*+n}(C_{G \times H}^*(X \times Y)), \quad x \mapsto x \times \text{Ind}([\mathcal{D}_Y])$$

are (rationally) split-injective.

Specializing to universal coverings of closed manifolds yields the desired generalization of Theorem 1.1:

Corollary 1.6. *Let N be a closed spin^c -manifold of dimension n such that its universal covering \tilde{N} is (rationally) Higson-essential. Then for every closed manifold M the maps*

$$S_*^G(\tilde{M}) \rightarrow S_{*+n}^{G \times H}(\tilde{M} \times \tilde{N}), \quad x \mapsto x \times [\mathcal{D}_N],$$

and

$$K_*(C_{\text{red}}^*(G)) \rightarrow K_{*+n}(C_{\text{red}}^*(G \times H)), \quad x \mapsto x \times \text{Ind}_H([\mathcal{D}_N])$$

where $G = \pi_1 M$ and $H = \pi_1 N$, are (rationally) split-injective.

The existence of a γ -element has a far stronger consequences than this. Traditionally, this concept appeared in proofs of the strong Novikov conjecture. Indeed, it implies injectivity of the entire equivariant index map—not just non-vanishing of the index of the fundamental class. Using an equivariant version of the co-assembly map it also enables us to prove a stronger result in the case of aspherical complexes.

Theorem 1.7 (see Corollary 6.5). *Let N be a finite aspherical complex, and assume that $H = \pi_1 N$ has a γ -element.*

Then for every proper metric space X endowed with a proper action of a countable discrete group G , the external product maps

$$S_m^G(X) \otimes K_n(N) \rightarrow S_{m+n}^{G \times H}(X \times \tilde{N})$$

and

$$K_m(C_G^* X) \otimes K_n(N) \rightarrow K_{m+n}(C_{G \times H}^*(X \times \tilde{N}))$$

are rationally injective for each $m, n \in \mathbb{Z}$.

Corollary 1.8. *In the setup of Theorem 1.7, where $X = \tilde{M}$ is the universal covering of a closed manifold M and $G = \pi_1 M$, the external product maps*

$$S_m^G(\tilde{M}) \otimes K_n(N) \rightarrow S_{m+n}^{G \times H}(\tilde{M} \times \tilde{N}) \tag{1.3}$$

and

$$K_m(C_{\text{red}}^* G) \otimes K_n(N) \rightarrow K_{m+n}(C_{\text{red}}^*(G \times H)), \tag{1.4}$$

are rationally injective for each $m, n \in \mathbb{Z}$.

One might conjecture that in the case of an aspherical manifold whose universal covering is stably hyperEuclidean, the rational version of Theorem 1.1 should also follow from Corollary 1.8. This would be the case if the following open question had a positive answer.

Question 1.9. *Let M be a closed aspherical manifold. If M is stably hyperEuclidean, does the fundamental group $\pi_1 M$ admit a γ -element?*

If we assume that $\pi_1 M$ does not only have a γ -element, but that $\pi_1 M$ even satisfies the Baum–Connes conjecture, then the rational injectivity of (1.4) upgrades to rational bijectivity due to the Künneth formula [CEOO04]. Our next result is that under the assumption of Baum–Connes we also get such an upgrade for (1.3). To formulate this result in its full generality, we introduce the following notation for the *representable K-homology* and its counterpart for the structure group:

$$\mathrm{RK}_*^G(X) := \varinjlim_K \mathrm{K}_*^G(K), \quad (1.5)$$

$$\mathrm{RS}_*^G(X) := \varinjlim_K \mathrm{S}_*^G(K), \quad (1.6)$$

where the colimits run over G -cocompact subsets $K \subseteq X$. Note that (1.1) then induces a sequence

$$\cdots \rightarrow \mathrm{K}_{*+1}(\mathrm{C}_{\mathrm{red}}^*G) \xrightarrow{\partial} \mathrm{RS}_*^G(X) \rightarrow \mathrm{RK}_*^G(X) \xrightarrow{\mathrm{Ind}} \mathrm{K}_*(\mathrm{C}_{\mathrm{red}}^*G) \rightarrow \cdots. \quad (1.7)$$

Theorem 1.10 (Theorem 7.1). *Let H be a countable discrete group. Assume that H is torsion-free and satisfies the Baum–Connes conjecture for all coefficient C^* -algebras with trivial H -action.³*

Then for any simplicial complex M , the external product map

$$\mathrm{RS}_*^G(\widetilde{M}) \otimes \mathrm{RK}_*(BH) \rightarrow \mathrm{RS}_*^{G \times H}(\widetilde{M} \times EH),$$

where $G = \pi_1 M$, is rationally an isomorphism. If $\mathrm{RK}_(BH)$ is torsion-free, then it is integrally an isomorphism.*

1.1 Slant products

The main technical innovation of our paper is the construction of *slant products* between the various groups which appear in the Higson–Roe sequence and the K-theory of stable Higson corona. The most general of these incorporate proper actions of countable discrete groups on all involved spaces as is needed for the proof of Theorem 1.7. However, for simplicity, start with an exposition of the non-compact setting ignoring all group actions. The next theorem and the following properties is a summary of Section 4.

Theorem 1.11. *Let X be a proper metric space, and let Y be a proper metric space of bounded geometry.*

For each element $\theta \in \mathrm{K}_{1-q}(\mathfrak{c}^{\mathrm{red}}Y)$ we construct natural slant products $/\theta$ such that we have a commuting diagram

$$\begin{array}{ccccccc} \mathrm{S}_p(X \times Y) & \longrightarrow & \mathrm{K}_p(X \times Y) & \longrightarrow & \mathrm{K}_p(\mathrm{C}^*(X \times Y)) & \xrightarrow{\partial} & \mathrm{S}_{p-1}(X \times Y) \\ \downarrow / \theta & & \downarrow / \theta & & \downarrow / \theta & & \downarrow / \theta \\ \mathrm{S}_{p-q}(X) & \longrightarrow & \mathrm{K}_{p-q}(X) & \longrightarrow & \mathrm{K}_{p-q}(\mathrm{C}^*X) & \xrightarrow{\partial} & \mathrm{S}_{p-1-q}(X) \end{array} \quad (1.8)$$

and such that the slant products have the properties listed below in Properties 1.13.

We recall the relevant definition of bounded geometry at the beginning of Section 4.1, and the stable Higson corona $\mathfrak{c}^{\mathrm{red}}Y$ and the corresponding coarse co-assembly map are recalled at the beginning of Section 4.

³For example, H could be a-T-menable [HK01] or it could be hyperbolic [Laf12; Pus12].

Remark 1.12 (Notation). To state certain results more concisely, we will use the symbol \mathcal{HR} as a generic placeholder for any constituent of the Higson–Roe sequence. That is, $\mathcal{HR}_*(X)$ can stand for either $S_*(X)$, $K_*(X)$ or $K_*(C^*X)$. \blacklozenge

Properties 1.13. The slant products in Theorem 1.11 satisfy the following properties:

- (i) If Y has continuously bounded geometry, then the slant product

$$/\theta: K_p(X \times Y) \rightarrow K_{p-q}(X)$$

is compatible with coarse co-assembly $\mu^*: K_{1-q}(\mathbf{c}^{\text{red}}Y) \rightarrow K^q(Y)$, that is, for all $x \in K_p(X \times Y)$ we have

$$x/\theta = x/\mu^*(\theta),$$

where the slant product on the right hand side is the usual slant product of locally finite K-homology with compactly supported K-theory.

- (ii) For any element $y \in K_q(C^*Y)$ the composition⁴

$$K_p(C^*X) \xrightarrow{\times y} K_{p+q}(C^*(X \times Y)) \xrightarrow{/\theta} K_p(C^*X)$$

equals multiplication with $\langle y, \theta \rangle \in \mathbb{Z}$ on $K_p(C^*X)$, where this is the pairing constructed by Emerson and Meyer [EM06, Section 6], possibly up to a sign $(-1)^q$.⁵

Furthermore, for $y \in K_q(Y)$ the compositions

$$\begin{aligned} K_p(X) &\xrightarrow{\times y} K_{p+q}(X \times Y) \xrightarrow{/\theta} K_p(X) \\ S_p(X) &\xrightarrow{\times y} S_{p+q}(X \times Y) \xrightarrow{/\theta} K_p(X) \end{aligned}$$

equal multiplication by $\langle y, \mu^*(\theta) \rangle \in \mathbb{Z}$, where this is the usual pairing between locally finite K-homology and compactly supported K-theory.

- (iii) Denote by $\beta \in K_{1-n}(\mathbf{c}^{\text{red}}\mathbb{R}^n)$ the class corresponding to the Bott element of Euclidean space, that is, $\mu^*(\beta) \in K^n(\mathbb{R}^n) \cong \mathbb{Z}$ is the generator.

Then the slant product $/\beta: \mathcal{HR}_{p+n}(X \times \mathbb{R}^n) \rightarrow \mathcal{HR}_p(X)$ is an isomorphism and coincides with n -fold suspension isomorphism⁶.

- (iv) The slant products are compatible with the coarsification and co-coarsification maps⁷.

Recall that the coarse assembly map $\mu: K_p(Z) \rightarrow K_p(C^*(Z))$ factors as

$$K_p(Z) \xrightarrow{c} KX_p(Z) \xrightarrow{\mu} K_p(C^*(Z))$$

and that the coarse co-assembly map $\mu^*: K_{1-q}(\mathbf{c}^{\text{red}}Z) \rightarrow K^q(Z)$ as

$$\mu^*: K_{1-q}(\mathbf{c}^{\text{red}}Z) \xrightarrow{\mu^*} KX^q(Z) \xrightarrow{c^*} K^q(Z).$$

⁴The construction of the external product $- \times y$ is recalled in Section 3.

⁵Emerson and Meyer did not specify their sign convention for the pairing.

⁶Or in other words, the Mayer–Vietoris boundary map associated to the (coarsely) excisive cover $X \times \mathbb{R} = X \times (-\infty, 0] \cup X \times [0, \infty)$

⁷The co-coarsification maps are sometimes called “character maps” in the literature.

We have slant products on the coarsification of locally finite K-homology such that the diagram

$$\begin{array}{ccccc} K_p(X \times Y) & \xrightarrow{c} & KX_p(X \times Y) & \xrightarrow{\mu} & K_p(C^*(X \times Y)) \\ \downarrow / \theta & & \downarrow / \theta & & \downarrow / \theta \\ K_{p-q}(X) & \xrightarrow{c} & KX_{p-q}(X) & \xrightarrow{\mu} & K_{p-q}(C^*X) \end{array}$$

commutes and such that for $x \in KX_p(X \times Y)$ we have $x/\theta = x/\mu^*(\theta)$. The slant product on the right hand side is between the coarsifications of locally finite K-homology and of compactly supported K-theory (see Definition 4.45). \blacklozenge

Property (i) in combination with commutativity of the middle square in the diagram (1.8) translates to the formula

$$\mu(x/\mu^*(\theta)) = \mu(x)/\theta$$

for all $x \in K_p(X \times Y)$ and all $\theta \in K_{1-q}(c^{\text{red}}Y)$.

As an example, we mention the following result which can be obtained from the existence of the slant products with the above properties:

Corollary 1.14 (Corollary 6.4). *Let Y be either*

- (i) *a uniformly contractible, proper metric space of continuously bounded geometry which is scaleable,*
- (ii) *a uniformly contractible, proper metric space of continuously bounded geometry which admits an expanding and coherent combing, or*
- (iii) *the universal cover EG of the classifying space BG of a group G , if BG is a finite complex and G is coarsely embeddable into a Hilbert space.*

Then for every proper metric space Y the external product map

$$\mathcal{HR}_m(X) \otimes K_n(Y) \rightarrow \mathcal{HR}_{m+n}(X \times Y)$$

is rationally injective. Here we used the notation from Remark 1.12.

1.2 Equivariant slant products

As previously mentioned, there is an equivariant version of our slant products. We exhibit this construction in Section 5. It requires an equivariant version of the coarse co-assembly map. If Y is endowed with a proper action of a countable discrete group H , we obtain an induced action on the stable Higson corona. Using this, one obtains a co-assembly map of the form

$$\mu_H^*: K_{1-*}(c^{\text{red}}Y \rtimes_{\mu} H) \rightarrow K_H^*(Y),$$

where, in general, μ can be any exact crossed product functor in the sense of [BGW16, Def. 3.1], or the reduced one if H is exact. Note that this is a different version of the equivariant co-assembly map than the one considered by Emerson and Meyer [EM07; EM08]. We discuss this and related questions around exactness in Sections 5.1 and 5.2. In the following, we assume that an appropriate choice for μ has been fixed.

Theorem 1.15. *Let X and Y be proper metric spaces, where Y has bounded geometry, which are endowed with proper isometric actions of countable discrete groups G and H , respectively. Then for each element $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Y \rtimes_{\mu} H)$, we construct natural slant products $/\theta$ such that we have a commuting diagram*

$$\begin{array}{ccccccc} S_p^{G \times H}(X \times Y) & \longrightarrow & K_p^{G \times H}(X \times Y) & \longrightarrow & K_p(C_{G \times H}^*(X \times Y)) & \xrightarrow{\partial} & S_{p-1}^{G \times H}(X \times Y) \\ \downarrow / \theta & & \downarrow / \theta & & \downarrow / \theta & & \downarrow / \theta \\ S_{p-q}^G(X) & \longrightarrow & K_{p-q}^G(X) & \longrightarrow & K_{p-q}(C_G^*X) & \xrightarrow{\partial} & K_{p-1-q}^G(X). \end{array}$$

The equivariant slant products satisfy formal properties analogous to Properties 1.13. We refer to Section 5.4 for the details. Moreover, in Section 5.5 we prove that for free actions our equivariant slant product on K-homology is identified with the usual slant product on K-homology of the quotient space up to canonical induction isomorphisms.

The equivariant slant products are applied to prove the following injectivity result for the external products on the Higson–Roe analytic sequence (1.2). In the following theorem we write \tilde{M} for the universal cover of M , and we denote $G := \pi_1(M)$.

Theorem 1.16 (see Theorem 6.1). *Let N be a finite complex, $H = \pi_1 N$, and let $z \in K_n(N)$. Assume that there is $\theta \in K_{1-n}(\mathfrak{c}^{\text{red}}\tilde{N} \rtimes_{\mu} H)$ with $\langle z, \mu_H^*(\theta) \rangle = 1$ (or $\langle z, \mu_H^*(\theta) \rangle \neq 0$, respectively).*

Then for every finite complex M , all vertical arrows in the following diagram (with $G = \pi_1 M$) are split-injective (rationally split-injective, respectively).

$$\begin{array}{ccccccc} K_{*+1}(C_{\text{red}}^*G) & \xrightarrow{\partial} & S_*^G(\tilde{M}) & \longrightarrow & K_*(M) & \xrightarrow{\text{Ind}_G} & K_*(C_{\text{red}}^*G) \\ \downarrow \times \text{Ind}_H(z) & & \downarrow \times z & & \downarrow \times z & & \downarrow \times \text{Ind}_H(z) \\ K_{*+1+n}(C_{\text{red}}^*(G \times H)) & \xrightarrow{\partial} & S_{*+n}^{G \times H}(\tilde{M} \times \tilde{N}) & \longrightarrow & K_{*+n}(M \times N) & \xrightarrow{\text{Ind}_{G \times H}} & K_{*+n}(C_{\text{red}}^*(G \times H)) \end{array}$$

Indeed, a right inverse for the external product maps from the theorem is given by our equivariant slant products with the class θ . A similar result was proved by Zenobi [Zen17, Rem. 5.20].

Using a slightly more sophisticated version of the above theorem (see Theorem 6.2), we can deduce Theorem 1.7. This is because the existence of a γ -element implies surjectivity of the equivariant coarse co-assembly map for aspherical complexes (see Corollary 5.3).

1.3 Geometric applications

1.3.1 The Stolz sequence for positive scalar curvature

The Stolz sequence for positive scalar curvature [Sto98] is a sequence of bordism groups incorporating Riemannian metrics of positive scalar curvature (psc). It is in some sense analogous to the surgery sequence from geometric topology. Given a closed smooth manifold M , we will denote by $\mathcal{R}^+(M)$ the set of Riemannian metrics of positive scalar curvature on M .

The Stolz sequence has received considerable attention in higher index theory starting with work of Piazza and Schick [PS14] and Xie and Yu [XY14] who established that it admits a map to the analytic sequence of Higson and Roe. We recall the result here in the case of classifying spaces of groups. Here it is convenient to use (1.7).

Theorem 1.17 ([PS14; XY14]). *Let G be a countable discrete group. There is a commutative diagram of exact sequences taking Stolz' positive scalar curvature sequence to the analytic sequence of Higson and Roe.*

$$\begin{array}{ccccccccc}
\Omega_m^{\text{spin}}(\text{BG}) & \longrightarrow & \text{R}_m^{\text{spin}}(\text{BG}) & \xrightarrow{\partial} & \text{Pos}_{m-1}^{\text{spin}}(\text{BG}) & \longrightarrow & \Omega_{m-1}^{\text{spin}}(\text{BG}) & \longrightarrow & \text{R}_{m-1}^{\text{spin}}(\text{BG}) \\
\downarrow \beta & & \downarrow \alpha & & \downarrow \rho & & \downarrow \beta & & \downarrow \alpha \\
\text{RK}_m(\text{BG}) & \xrightarrow{\mu} & \text{K}_m(\text{C}_{\text{red}}^*G) & \xrightarrow{\partial} & \text{RS}_{m-1}^G(\text{EG}) & \longrightarrow & \text{RK}_{m-1}(\text{BG}) & \xrightarrow{\mu} & \text{K}_{m-1}(\text{C}_{\text{red}}^*G)
\end{array}$$

We briefly explain the constituents of the top sequence above. First, $\Omega_*^{\text{spin}}(\text{BG})$ is the usual spin bordism group of BG . The group $\text{R}_*^{\text{spin}}(\text{BG})$ consists of bordism classes of pairs (W, g) , where W is a compact spin manifold together with a continuous map $W \rightarrow \text{BG}$ and $g \in \mathcal{R}^+(\partial W)$. Finally, the *positive scalar curvature bordism group* $\text{Pos}_*^{\text{spin}}(\text{BG})$ consists of bordism classes of pairs (M, g) , where M is a closed spin manifold together with a continuous map $M \rightarrow \text{BG}$ and $g \in \mathcal{R}^+(M)$. The map β is the Atiyah–Bott–Shapiro orientation, α is the *higher relative index* and ρ is the *higher ρ -invariant*.

The positive scalar curvature bordism group admits an external product with the spin bordism group as follows.

$$\begin{aligned}
\text{Pos}_m^{\text{spin}}(\text{BG}) \otimes \Omega_n^{\text{spin}}(\text{BH}) &\rightarrow \text{Pos}_{m+n}^{\text{spin}}(\text{B}(G \times H)) \\
[M, g] \otimes [N] &\mapsto [M \times N, g \oplus g_N]
\end{aligned}$$

Here we choose any Riemannian metric g_N on N such that the product metric $g \oplus g_N$ still has positive scalar curvature (this can always be achieved by rescaling). Any two such choices yield isotopic metrics on the product (again by a rescaling argument) and in particular the same bordism class. By the product formula for the higher ρ -invariant (see [Zei16a]), it is compatible with the external product for the analytic structure group. That is, the following diagram commutes.

$$\begin{array}{ccc}
\text{Pos}_m^{\text{spin}}(\text{BG}) \otimes \Omega_n^{\text{spin}}(\text{BH}) & \longrightarrow & \text{Pos}_{m+n}^{\text{spin}}(\text{B}(G \times H)) \\
\rho \otimes \beta \downarrow & & \rho \downarrow \\
\text{RS}_m^G(\text{EG}) \otimes \text{RK}_n(\text{BH}) & \longrightarrow & \text{RS}_{m+n}^{G \times H}(\text{E}(G \times H))
\end{array}$$

Similarly, we have a diagram involving the relative group.

$$\begin{array}{ccc}
\text{R}_m^{\text{spin}}(\text{BG}) \otimes \Omega_n^{\text{spin}}(\text{BH}) & \longrightarrow & \text{R}_{m+n}^{\text{spin}}(\text{B}(G \times H)) \\
\alpha \otimes \beta \downarrow & & \alpha \downarrow \\
\text{K}_m(\text{C}_{\text{red}}^*\Gamma) \otimes \text{RK}_n(\text{BH}) & \longrightarrow & \text{K}_{m+n}(\text{C}_{\text{red}}^*(G \times H))
\end{array}$$

We now obtain the following corollary as a consequence of Theorem 1.5. In the following we say that a spin manifold is Higson-essential if it is with respect to the induced spin^c -structure.

Corollary 1.18. *Let N be a closed spin manifold with $\pi_1 N = H$ such that its universal covering \tilde{N} is Higson-essential. Let $[M_i, g_i] \in \text{Pos}_m^{\text{spin}}(\text{BG})$ with $\rho(M_0, g_0) \neq \rho(M_1, g_1) \in \text{RS}_m^G(\text{EG})$. Then also*

$$\rho(M_0 \times N, g_0 \oplus g_N) \neq \rho(M_1 \times N, g_1 \oplus g_N) \in \text{RS}_{m+n}^{G \times H}(\text{E}(G \times H)).$$

In particular, $(M_0 \times N, g_0 \oplus g_N)$ and $(M_1 \times N, g_1 \oplus g_N)$ represent different bordism classes in $\text{Pos}_{m+n}^{\text{spin}}(\text{B}(G \times H))$.

In [Zei16a, Corollary 6.10] this corollary was formulated for N aspherical with $H = \pi_1 N$ of finite asymptotic dimension. This was based on Dranishnikov's theorem that the universal covering of such a manifold is stably hypereuclidean [Dra06]. Our present method strictly improves Zeidler's result, because if N is aspherical and $H = \pi_1 N$ admits a coarse embedding into Hilbert space (which is far more general than having finite asymptotic dimension), it satisfies the hypothesis of the corollary above (compare Remark 6.8 and Corollary 6.4 (c)).

Furthermore, if M is a closed spin manifold with $\pi_1 M = G$ and $g_0, g_1 \in \mathcal{R}^+(M)$, then $(M \times [0, 1], (g_0, g_1))$ represents a class in $R_{m+1}^{\text{spin}}(BG)$, where $(g_0, g_1) \in \mathcal{R}^+(\partial(M \times [0, 1])) = \mathcal{R}^+(M) \times \mathcal{R}^+(M)$. The corresponding relative index class in $K_{m+1}(C_{\text{red}}^*G)$ is the (*higher*) *index difference* of g_0 and g_1 , denoted by $\text{inndiff}_G(g_0, g_1) \in K_{m+1}(C_{\text{red}}^*G)$. If the index difference is non-zero, then the two metrics are *not concordant* as positive scalar curvature metrics. In particular, they are not isotopic as psc metrics.

Again we obtain the following from Theorem 1.5.

Corollary 1.19. *Let N be a closed spin manifold with $\pi_1 N = H$ such that its universal covering \tilde{N} is Higson-essential. Let M be a closed spin manifold with $\pi_1 M = G$ and $g_0, g_1 \in \mathcal{R}^+(M)$ such that $\text{inndiff}_G(g_0, g_1) \neq 0 \in K_{m+1}(C_{\text{red}}^*G)$. Then*

$$\text{inndiff}_{G \times H}(g_0 \oplus g_N, g_1 \oplus g_N) \neq 0 \in K_{m+n+1}(C_{\text{red}}^*(G \times H)).$$

In particular, $g_0 \oplus g_N$ and $g_1 \oplus g_N$ are not concordant as positive scalar curvature metrics on $M \times N$.

The final conclusion of the previous corollary is also true for almost-spin manifolds N , see Corollary 1.23 below.

Examples of pairs of (M_i, g_i) with different ρ -invariants or pairs of metrics with non-trivial index-differences exist in abundance. For the ρ -invariants, such examples always exist if the group G satisfies the strong Novikov conjecture and has torsion. For the existence of non-trivial index differences, this is also true for torsion-free groups. In fact, under the Novikov assumption, lower bounds on the ranks of the groups $\text{Pos}_*^{\text{spin}}(BG)$ and $R_*^{\text{spin}}(BG)$ can be given in terms of the homology of G , see for instance [WY15; BZ18; ERW19; XYZ17]. These results are always proved by showing that secondary index maps such as α and ρ have a large range.

We deduce from Theorem 1.7 (applying it inside the relevant colimits) that the size of the range is preserved under certain products.

Corollary 1.20. *Suppose the group H has a γ -element and admits a finite model for BH .*

- (i) *Let $V \subseteq \text{Pos}_m^{\text{spin}}(BG)$ be a subset such that $\rho(V) \subseteq \text{RS}_m^G(EG)$ generates a subgroup of rank $\geq k$. Then $V \otimes \Omega_n^{\text{spin}}(BH)$ generates a subgroup of rank*

$$\geq k \cdot \text{rank}(\beta\Omega_n^{\text{spin}}(BH))$$

in $\text{Pos}_{m+n}^{\text{spin}}(B(G \times H))$.

- (ii) *Let $V \subseteq R_m^{\text{spin}}(BG)$ be a subset such that $\alpha(V) \subseteq K_m(C_{\text{red}}^*G)$ generates a subgroup of rank $\geq k$. Then $V \otimes \Omega_n^{\text{spin}}(BH)$ generates a subgroup of rank*

$$\geq k \cdot \text{rank}(\beta\Omega_n^{\text{spin}}(BH))$$

in $R_{m+n}^{\text{spin}}(B(G \times H))$.

1.3.2 Positive scalar curvature on non-compact manifolds

As our methods deal with non-compact spaces, there are applications to uniform positive scalar curvature on non-compact manifolds. However, here it is necessary to restrict the large-scale type of the metrics which we consider. Otherwise, the fact that \mathbb{R}^n for $n \geq 3$ admits metrics of uniform positive scalar curvature would lead to counterexamples to the kind of results we have in mind. We use a similar setup as in [Zei16b, Section 1.3].

Let X be a spin n -manifold and g_X a fixed complete Riemannian metric on X . We let $\mathcal{R}(X, g_X)$ denote the set of all those Riemannian metrics g on X such that the identity map $(X, d_g) \rightarrow (X, d_{g_X})$ is uniformly continuous. Note that each metric in $\mathcal{R}(X, g_X)$ is automatically complete because g_X is. Moreover, the identity map $(X, d_g) \rightarrow (X, d_{g_X})$ is coarse because (X, d_g) is a length space and so uniformly continuous maps are automatically large-scale Lipschitz. Moreover, we let $\mathcal{R}^+(X, g_X)$ be the set of those metrics in $\mathcal{R}(X, g_X)$ with uniformly positive scalar curvature. If X is furnished with a proper isometric action of a countable discrete group G and g_X is G -invariant, we write $\mathcal{R}(X, g_X)^G$ and $\mathcal{R}^+(X, g_X)^G$ for the corresponding subsets of G -invariant metrics. Note that if $\text{Ind}([\not{D}_X]) \neq 0 \in \mathbf{K}_m(\mathbf{C}_G^*X)$, where we define the Roe algebra with respect to the metric g_X , then $\mathcal{R}^+(X, g_X)^G = \emptyset$.

Definition 1.21. We say that two metrics $g_0, g_1 \in \mathcal{R}^+(X, g_X)^G$ are *concordant* if there exists a metric in $\mathcal{R}^+(X \times \mathbb{R}, g_X \oplus dt^2)^G$ which restricts to $g_0 \oplus dt^2$ on $X \times (-\infty, 0]$ and to $g_1 \oplus dt^2$ on $X \times [1, \infty)$. \blacklozenge

Given $g_0, g_1 \in \mathcal{R}^+(X, g_X)^G$, there is the (*equivariant*) *coarse index difference* $\text{inddiff}(g_0, g_1) \in \mathbf{K}_{n+1}(\mathbf{C}_G^*X)$ which vanishes if g_0 and g_1 are concordant, see [Zei16b, Section 2.2.4], [Zei16a, Section 4.4]. Note that if $X = \widetilde{M}$ is the universal covering of a closed manifold M , then the equivariant coarse index difference agrees with the index difference considered in the previous subsection via the identification $\mathbf{K}_*(\mathbf{C}_G^*(\widetilde{M})) = \mathbf{K}_*(\mathbf{C}_{\text{red}}^*G)$. Moreover, given another complete spin manifold (Y, g_Y) such that $g_i \oplus g_Y$ has uniformly positive scalar curvature for $i = 0, 1$, then the coarse index difference satisfies the product formula

$$\text{inddiff}(g_0 \oplus g_Y, g_1 \oplus g_Y) = \text{inddiff}(g_0, g_1) \times \text{Ind}([\not{D}_Y]).$$

Using our slant products, we obtain the following results.

Corollary 1.22. *Let (Y, g_Y) be a complete spin manifold of continuously bounded geometry which is Higson-essential. Let M be a closed spin manifold and $G := \pi_1 M$.*

- (i) *If $\text{Ind}_G([\not{D}_M]) \neq 0 \in \mathbf{K}_m(\mathbf{C}_{\text{red}}^*G)$, then $\mathcal{R}^+(M \times Y, g_M \oplus g_Y) = \emptyset$.*
- (ii) *Let $g_0, g_1 \in \mathcal{R}^+(M)$ such that $\text{inddiff}_G(g_0, g_1) \neq 0 \in \mathbf{K}_{m+1}(\mathbf{C}_{\text{red}}^*G)$. Then $g_0 \oplus g_Y$ and $g_1 \oplus g_Y$ are not concordant on $M \times Y$.*

Analogous statements apply if Y is rationally Higson-essential and the relevant index class is rationally non-zero.

Proof. Lifting metrics induces an identification

$$\mathcal{R}(M \times Y, g_M \oplus g_Y) = \mathcal{R}(\widetilde{M} \times Y, g_{\widetilde{M}} \oplus g_Y)^G,$$

preserving products, uniform psc and concordances. Hence it suffices to prove the corresponding statements in $\mathcal{R}^+(\widetilde{M} \times Y, g_{\widetilde{M}} \oplus g_Y)^G$. The product formula for the index and index difference together with Theorem 1.5 imply $\text{Ind}([\not{D}_{\widetilde{M} \times Y}]) \neq 0 \in \mathbf{K}_{m+n}(\mathbf{C}_G^*(\widetilde{M} \times Y))$ in part (i) and $\text{inddiff}(\widetilde{g}_0 \oplus g_Y, \widetilde{g}_1 \oplus g_Y) \neq 0 \in \mathbf{K}_{m+n+1}(\mathbf{C}_G^*(\widetilde{M} \times Y))$ in part (ii). \square

In particular, we deduce results for closed almost spin manifold (that is, the universal covering is spin but not necessarily the manifold itself).

Corollary 1.23. *Let N be a closed manifold with $\pi_1 N = H$ such that its universal covering \tilde{N} is spin and Higson-essential. Let M be a closed spin manifold with $\pi_1 M = G$.*

- (i) *If $\text{Ind}_G[\not{D}_M] \neq 0 \in K_m(C_{\text{red}}^*G)$, then $\mathcal{R}^+(M \times N) = \emptyset$.*
- (ii) *Let $g_0, g_1 \in \mathcal{R}^+(M)$ such that $\text{inndiff}_G(g_0, g_1) \neq 0 \in K_{m+1}(C_{\text{red}}^*G)$. Then $g_0 \oplus g_N$ and $g_1 \oplus g_N$ are not concordant as positive scalar curvature metrics on $M \times N$, where g_N is some Riemannian metric on N such that $g_i \oplus g_N \in \mathcal{R}^+(M \times N)$*

Proof. If $\mathcal{R}^+(M \times N) \neq \emptyset$, then lifting the metric shows $\mathcal{R}^+(M \times \tilde{N}, g_M \oplus \tilde{g}_N) \neq \emptyset$ for any choice of Riemannian metrics g_M and g_N on M and N , respectively. Hence the statement follows from Corollary 1.22 (i). The second part is reduced to Corollary 1.22 in an analogous fashion. \square

For $g \in \mathcal{R}^+(X, g_X)^G$, there is also a ρ -invariant $\rho(g) \in S_m^G(X)$. If $\rho(g_0) \neq \rho(g_1)$, then the metrics are also not concordant. Theorem 1.5 then implies an analogous version of Corollary 1.22 for the ρ -invariant. However, $\rho(g_0) \neq \rho(g_1)$ already implies $\text{inndiff}(g_0, g_1) \neq 0$. In fact, this ρ -invariant is a coarse bordism invariant in a suitable sense, see [Zei16b, Section 2.4.2], but formulating this requires some care to ensure that the structure groups on the two different ends of a bordism remain comparable. In the right setup, it is then also possible to establish a non-compact version of the final conclusion from Corollary 1.18, but we refrain from expounding the details here.

1.4 Generalizations and questions

1.4.1 Coefficients

Instead of working with the ordinary Roe algebra and ordinary stable Higson corona, we could have used throughout this paper their corresponding versions with coefficients, that is, Hilbert modules \mathcal{E} , \mathcal{F} and \mathcal{G} for the Roe algebras of the corresponding spaces and a C^* -algebra C for the stable Higson corona.⁸ K-homology and the analytic structure group can be similarly enriched.

For $a \in K_m(Y; \mathcal{F})$ we then have the following diagram for the external product by a :

$$\begin{array}{ccccccc} S_n(X; \mathcal{E}) & \longrightarrow & K_n(X; \mathcal{E}) & \longrightarrow & K_n(C^*(X; \mathcal{E})) & \longrightarrow & S_{n-1}(X; \mathcal{E}) \\ \downarrow \times a & & \downarrow \times a & & \downarrow \times \mu(a) & & \downarrow \times a \\ S_{m+n}(X \times Y; \mathcal{E} \otimes \mathcal{F}) & \longrightarrow & K_{m+n}(X \times Y; \mathcal{E} \otimes \mathcal{F}) & \longrightarrow & K_{m+n}(C^*(X \times Y; \mathcal{E} \otimes \mathcal{F})) & \longrightarrow & S_{m+n-1}(X \times Y; \mathcal{E} \otimes \mathcal{F}) \end{array}$$

and for $b \in K_{1-m}(\mathfrak{c}^{\text{red}}(Y; C))$ the following version of the slant products by b :

$$\begin{array}{ccccccc} S_n(X \times Y; \mathcal{G}) & \longrightarrow & K_n(X \times Y; \mathcal{G}) & \longrightarrow & K_n(C^*(X \times Y; \mathcal{G})) & \longrightarrow & S_{n-1}(X \times Y; \mathcal{G}) \\ \downarrow /b & & \downarrow / \mu^*(b) & & \downarrow /b & & \downarrow /b \\ S_{n-m}(X; \mathcal{G} \otimes C) & \longrightarrow & K_{n-m}(X; \mathcal{G} \otimes C) & \longrightarrow & K_{n-m}(C^*(X; \mathcal{G} \otimes C)) & \longrightarrow & S_{n-1-m}(X; \mathcal{G} \otimes C) \end{array}$$

⁸Roe algebras with coefficients in C^* -algebras were considered by Higson, Pedersen, and Roe [HPR97]. The definition of Roe algebras with coefficients in C^* -algebras given in [HPS15] is quickly seen to generalize to Hilbert modules; see also [Wul19].

Emerson and Meyer [EM06] defined the stable Higson corona from the very beginning with coefficients in C^* -algebras.

In particular, take $X = *$ to be a point and consider the standard Hilbert-module $\mathcal{A} = A \otimes \ell^2$, where A is some C^* -algebra. Then the Roe algebra $C^*(*; \mathcal{A})$ is isomorphic to $A \otimes \mathcal{K}(\ell^2)$ and so $K_*(C^*(*; \mathcal{A})) = K_*(A)$. In the situation of Theorem 1.5, we could then deduce (rational) injectivity of the external product map

$$K_*(A) \rightarrow K_{*+n}(A \otimes C_H^*(Y)), \quad x \mapsto x \times \text{Ind}[\not{D}_Y].$$

We do not discuss this any further in the paper to keep the notation lean.

1.4.2 Real K-theory

We have formulated the results of this paper in the framework of complex K-theory for simplicity. However, especially for applications to positive scalar curvature and spin geometry, it would be desirable to establish the analogous statements for KO-homology and the corresponding real version of the analytic structure group. Our construction of slant products is sufficiently abstract and does not use any idiosyncrasies of complex K-theory (such as—for instance—using 2-periodicity in an essential way). Hence the slant products also exist in the real setup and all applications that involve external products with a single element (such as Theorem 1.5) go through without essential change.

However, more care has to be taken with results that rely on universal coefficient and Künneth theorems (such as Theorems 1.7 and 1.10). They do not appear to readily generalize and instead would require a more elaborate framework such as in [Boe02].

1.4.3 Künneth sequence for the structure group

In Corollary 1.8, we could have also included the case of the external product for K-homology, that is, $K_m(M) \otimes K_n(N) \rightarrow K_{m+n}(M \times N)$. In fact, injectivity of this map holds in full generality due to the Künneth theorem for K-homology.

There is also a Künneth theorem for the K-theory of the reduced group C^* -algebras, provided the Baum–Connes conjecture is satisfied by one of the groups.⁹ Therefore, in this situation we get rational injectivity of (1.4) as in the Theorem 1.7. But the assumption of satisfying Baum–Connes is much stronger than admitting a γ -element.

It is now a natural question if there is also a version of the Künneth sequence for the structure group. For instance, one might ask if there is a short exact sequence of the form

$$0 \rightarrow (S_*^G(\widetilde{M}) \otimes K_*(N)) \oplus (K_*(M) \oplus S_*^H(\widetilde{N})) \rightarrow S_*^{G \times H}(\widetilde{M} \times \widetilde{N}) \rightarrow ?\text{Tor?} \rightarrow 0,$$

where $?\text{Tor?}$ is some suitable correction term analogous to the Tor-term in the Künneth sequence for K-homology. For this to make sense, it is probably necessary to diagonally divide out $S_*^G(\widetilde{M}) \otimes S_*^H(\widetilde{N})$ in the term on the left. If such a sequence exists, then it should ideally imply the conclusion of Theorem 1.10. However, by the result of Section 2, proving the existence of such a sequence will—realistically—require at least some hypotheses related to the strong Novikov conjecture.

⁹Tu [Tu99] proved that if G is amenable, then C_{red}^*G lies in the bootstrap class and hence satisfies the Künneth formula. That C_{red}^*G satisfies the Künneth formula if G satisfies the Baum–Connes conjecture was proven by Chabert, Echterhoff, and Oyono-Oyono [CEO04].

1.4.4 Groups with torsion

Many of our results (such as Theorem 1.7) involve finite classifying spaces of countable discrete groups (that is, finite aspherical complexes) which restricts them to torsion-free groups. However, in our general constructions (especially in Section 5) we need the groups to act only properly and not necessarily freely. Thus many of these results will have corresponding versions for groups with torsion if we work with the classifying space for proper actions $\underline{E}G$ instead—at least if the latter admits a G -finite model.

1.4.5 Exactness and the stable Higson corona

In Section 5, when introducing the equivariant versions of the slant products, we are working with crossed products $\mathfrak{c}^{\text{red}}Y \rtimes_{\mu} H$, where \rtimes_{μ} is any exact crossed product functor. This is necessary because the relevant co-assembly map might in general not exist for the reduced crossed product unless H is exact.

So the natural question arises in which situations $\mathfrak{c}^{\text{red}}Y$ is an amenable H - C^* -algebra (see Definition 5.7) and hence all different choices of crossed products for $\mathfrak{c}^{\text{red}}Y \rtimes H$ coincide. This is in general always the case when H is an amenable group, and we show that it furthermore holds in the case that H is a Gromov hyperbolic group acting on itself:

Proposition 1.24 (Example 5.13). *Let H be a Gromov hyperbolic group. Then $\mathfrak{c}^{\text{red}}H$ is an amenable H - C^* -algebra and $\mathfrak{c}^{\text{red}}H \rtimes_{\text{max}} H \cong \mathfrak{c}^{\text{red}}H \rtimes_{\text{red}} H$.*

One can ask about the concrete relation between amenability of $\mathfrak{c}^{\text{red}}H$, the equality $\mathfrak{c}^{\text{red}}H \rtimes_{\text{max}} H \cong \mathfrak{c}^{\text{red}}H \rtimes_{\text{red}} H$ and the exactness of H . In Section 5.2 we get first partial results on this, and our general conjecture is the following:

Conjecture 1.25. *Let Y be a proper metric space equipped with an isometric action of a countable discrete group H . Then the following conditions are equivalent:*

- (a) *The group H acts amenably on the Higson compactification of Y .*
- (b) *$\mathfrak{c}^{\text{red}}Y$ is an amenable H - C^* -algebra.*
- (c) *The group H is exact and we have $\mathfrak{c}^{\text{red}}Y \rtimes_{\text{max}} H \cong \mathfrak{c}^{\text{red}}Y \rtimes_{\text{red}} H$.*

Note that there is a related statement in the dual situation, that is, for the uniform Roe algebra: H is exact if and only if the uniform Roe algebra C_u^*H is exact [BNW07].

2 Injectivity implies non-vanishing index

In this short section, we provide a proof for our motivating observation which was stated as Theorem 1.3 in the introduction.

Theorem 2.1. *Let N be a finite complex and $z \in K_n(N)$ be such that for every closed manifold M the map $S_*^G(\widetilde{M}) \rightarrow S_{*+n}^{G \times H}(\widetilde{M} \times \widetilde{N})$, $x \mapsto x \times z$, where $G = \pi_1 M$ and $H = \pi_1 N$, is rationally injective. Then $\text{Ind}_H(z) \in K_n(C_{\text{red}}^*H)$ is rationally non-zero.*

Proof. Suppose by contraposition that rationally $\text{Ind}_H(z) = 0$. For every M , there is the following commutative diagram.

$$\begin{array}{ccc} K_{*+1}(\mathbb{C}_{\text{red}}^* G) & \xrightarrow{\partial_M} & S_*^G(\widetilde{M}) \\ \downarrow -\times \text{Ind}_H(z) & & \downarrow -\times z \\ K_{*+1+n}(\mathbb{C}_{\text{red}}^*(G \times H)) & \xrightarrow{\partial_{M \times N}} & S_{*+n}^{G \times H}(\widetilde{M} \times \widetilde{N}) \end{array}$$

A construction of this diagram is provided in Section 3.2.

Since the left vertical arrow is rationally zero, we conclude that the image of $\partial_M \otimes \mathbb{Q}$ is contained in the kernel of

$$(-\times z) \otimes \mathbb{Q}: S_*^G(\widetilde{M}) \otimes \mathbb{Q} \rightarrow S_{*+n}^{G \times H}(\widetilde{M} \times \widetilde{N}) \otimes \mathbb{Q}.$$

Thus to complete the proof it suffices to find a closed manifold M such that $\partial_M \otimes \mathbb{Q} \neq 0$. Indeed, this happens for instance if M is such that $G = \pi_1 M$ is a non-trivial finite group. This is folklore but we briefly explain it for the convenience of the reader.

Let G be finite. Then

$$K_i(\mathbb{C}_{\text{red}}^* G) = K_i(\mathbb{C}[G]) \cong \begin{cases} \mathbb{R}(G) & i = 0, \\ 0 & i = 1, \end{cases}$$

where $\mathbb{R}(G)$ denotes the Grothendieck group of finite-dimensional complex G -representations. Elementary character theory shows that the rank of the abelian group $\mathbb{R}(G)$ is the number of conjugacy classes of elements in G . Hence it is greater than one because G is non-trivial. Moreover, since the homology of a finite group is torsion in all positive degrees, it follows (for instance by an application of the Atiyah–Hirzebruch spectral sequence) that

$$\text{RK}_i(\text{BG}) \otimes \mathbb{Q} \cong \text{RK}_i(*) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & i = 0, \\ 0 & i = 1, \end{cases}$$

where the isomorphism is induced by mapping onto the point. Thus, the rational assembly map $\mathbb{Q} \cong K_0(\text{BG}) \rightarrow K_0(\mathbb{C}[G]) \otimes \mathbb{Q} \cong \mathbb{R}(G) \otimes \mathbb{Q}$ is not surjective for dimension reasons (in fact, the image is generated by the left-regular representation of G). Putting these observations together, we see that the rational Higson–Roe sequence of G collapses to the following short exact sequence (with all other terms vanishing):

$$0 \rightarrow \underbrace{\text{RK}_0(\text{BG}) \otimes \mathbb{Q}}_{\cong \mathbb{Q}} \rightarrow \underbrace{K_0(\mathbb{C}_{\text{red}}^* G) \otimes \mathbb{Q}}_{\cong \mathbb{R}(G) \otimes \mathbb{Q}} \xrightarrow{\partial_G \otimes \mathbb{Q}} \text{RS}_1^G(\text{EG}) \otimes \mathbb{Q} \rightarrow 0$$

Since the assembly map is not surjective, $\partial_G \otimes \mathbb{Q} \neq 0$. Finally, for any M with $\pi_1 M = G$, the boundary map ∂_G factors as

$$\partial_G: K_{*+1}(\mathbb{C}_{\text{red}}^* G) \xrightarrow{\partial_M} S_*^G(\widetilde{M}) \rightarrow \text{RS}_*^G(\text{EG}).$$

Hence $\partial_G \otimes \mathbb{Q} \neq 0$ implies $\partial_M \otimes \mathbb{Q} \neq 0$. \square

Remark 2.2. In fact, the proof shows that injectivity of the external product map $S_*^G(\widetilde{M}) \xrightarrow{-\times z} S_{*+n}^{G \times H}(\widetilde{M} \times \widetilde{N})$ for any M , where $\pi_1 M = G$ is a non-trivial finite group, suffices to obtain the conclusion of the theorem. \blacklozenge

3 External products

In this section we will revisit the construction of the external product maps via localization algebras. The idea for this approach appeared first in work of Xie and Yu [XY14, Section 2.3.3] and was fleshed out by Zeidler [Zei16a, Section 3.3]. See also [WY19, Section 9.2]. We start in the setup without group actions and discuss the equivariant situation thereafter.

3.1 Non-equivariant case

Let X be a proper metric space. An X -module is a separable Hilbert space H_X endowed with a non-degenerate $*$ -representation $\rho_X: C_0(X) \rightarrow \mathcal{B}(H_X)$. Given an X module, the *Roe algebra* $C^*(\rho_X)$ is defined as the sub- C^* -algebra of $\mathcal{B}(H_X)$ generated by all locally compact operators of finite propagation. These notions are defined as follows.

- An operator $T \in \mathcal{B}(H_X)$ is called *locally compact*, if for every $f \in C_0(X)$ the operators $\rho_X(f)T$ and $T\rho_X(f)$ are compact operators.
- An operator $T \in \mathcal{B}(H_X)$ is said to have *finite propagation* if there exists an $R > 0$ such that $\rho_X(f)T\rho_X(g) = 0$ whenever the supports of f and g are further apart from each other than R . In this case, the propagation is said to be bounded by R .

An X -module (H_X, ρ_X) is called *ample*, if no non-zero function from $C_0(X)$ acts by a compact operator. From now on we shall assume that we have fixed an ample X -module (H_X, ρ_X) .

The crucial fact about Roe algebras is that their K-theory is independent of the choice of ample X -module [HR00b, Corollary 6.3.13] [WY19, Theorem 5.1.15] up to canonical isomorphism. Hence we usually suppress it from the notation by writing C^*X instead of $C^*(\rho_X)$. Nevertheless, later on we will have to consider the Roe algebras of X associated to different representations at once. In those cases, we default to the notation $C^*(\rho_X)$.

Further, one defines the *localization algebra* C_L^*X as the sub- C^* -algebra of $C_b([1, \infty), C^*X)$ generated by the bounded and uniformly continuous functions $L: [1, \infty) \rightarrow C^*X$ such that the propagation of $L(t)$ is finite for all $t \geq 1$ and tends to zero as $t \rightarrow \infty$. If it is constructed using an ample X -module, its K-theory is canonically isomorphic to the locally finite K-homology of the space X , that is,

$$K_*(C_L^*X) \cong KK_*(C_0(X), \mathbb{C}) \cong K_*(X).$$

This fact was originally established by Yu [Yu97] for finite complexes and was generalized to proper metric spaces by Qiao and Roe [QR10]. See also [WY19, Chap's. 6–7] for a self-contained development of analytic K-homology based on localization algebras.¹⁰ Related results that describe more general KK-groups using different versions of the localization algebra can be found in [DWW18].

Finally, the ideal in C_L^*X consisting of all such functions L with $L(1) = 0$ is denoted by $C_{L,0}^*X$. Its K-theory is called the *analytic structure group of X* and denoted $S_*(X) := K_*(C_{L,0}^*X)$.

¹⁰Note, however, that the localization algebras considered in [WY19] are slightly bigger and have somewhat better functoriality properties than the original versions. We use the original version in this paper because it is more convenient for our subsequent construction of slant products at the cost of a slightly more awkward approach to functoriality, see Remark 4.32.

These three C^* -algebras fit into a short exact sequence

$$0 \rightarrow C_{L,0}^*X \rightarrow C_L^*X \xrightarrow{\text{ev}_1} C^*X \rightarrow 0$$

and the induced long exact sequence is the *Higson–Roe sequence*

$$\cdots \rightarrow K_{*+1}(C^*X) \rightarrow S_*(X) \rightarrow K_*(X) \xrightarrow{\text{Ind}} K_*(C^*X) \rightarrow \cdots$$

with the map $\text{Ind} = (\text{ev}_1)_*$ induced by evaluation at 1 being the index map.

Given another proper metric space Y , we consider the above-mentioned C^* -algebras of Y and $X \times Y$ associated to a chosen ample representation $\rho_Y: C_0(Y) \rightarrow \mathcal{B}(H_Y)$ and the corresponding tensor product representation $\rho_{X \times Y}: C_0(X \times Y) \rightarrow \mathcal{B}(H_{X \times Y})$ on $H_{X \times Y} := H_X \otimes H_Y$ which is again ample.

Note that the tensor product of two locally compact operators of finite propagation is again locally compact and of finite propagation. Hence

$$C^*X \otimes C^*Y \subset C^*(X \times Y) \quad (3.1)$$

and this inclusion induces the external product

$$\times: K_m(C^*X) \otimes K_n(C^*Y) \rightarrow K_{m+n}(C^*(X \times Y)).$$

If L_1 and L_2 are functions in the generating subset of C_L^*X , C_L^*Y , respectively, then one readily verifies that the function

$$L: [1, \infty) \rightarrow C^*(X \times Y), \quad t \mapsto L_1(t) \otimes L_2(t)$$

also satisfies the propagation condition and yields an element of $C_L^*(X \times Y)$. If $L_1 \in C_{L,0}^*X$, then we will have $L \in C_{L,0}^*(X \times Y)$. This gives rise to isometric $*$ -homomorphisms

$$C_L^*X \otimes C_L^*Y \rightarrow C_L^*(X \times Y) \quad (3.2)$$

$$C_{L,0}^*X \otimes C_L^*Y \rightarrow C_{L,0}^*(X \times Y) \quad (3.3)$$

Remark 3.1. The fact that this works with the minimal tensor product on the left hand side can be seen as follows.

There are canonical faithful representations of C_L^*X , C_L^*Y and $C_L^*(X \times Y)$ on the Hilbert spaces $\ell^2([1, \infty)) \otimes H_X$, $\ell^2([1, \infty)) \otimes H_Y$ and $\ell^2([1, \infty)) \otimes H_{X \times Y}$, respectively. By the definition of the minimal tensor product we can thus see $C_L^*X \otimes C_L^*Y$ as a sub- C^* -algebra of $\mathcal{B}(\ell^2([1, \infty)) \otimes H_X \otimes \ell^2([1, \infty)) \otimes H_Y)$. Now, conjugation by the Hilbert space projection

$$\ell^2([1, \infty)) \otimes H_X \otimes \ell^2([1, \infty)) \otimes H_Y \rightarrow \ell^2([1, \infty)) \otimes H_{X \times Y}$$

onto the diagonal of $[1, \infty) \times [1, \infty)$ is a continuous linear map

$$\mathcal{B}(\ell^2([1, \infty)) \otimes H_X \otimes \ell^2([1, \infty)) \otimes H_Y) \rightarrow \mathcal{B}(\ell^2([1, \infty)) \otimes H_{X \times Y})$$

which is not a $*$ -homomorphism. But its restriction to the minimal tensor product $C_L^*X \otimes C_L^*Y$ is an isometric $*$ -homomorphism and has image contained in the canonically embedded sub- C^* -algebra $C_L^*(X \times Y)$, as can be seen on generators as above. \blacklozenge

The two $*$ -homomorphisms (3.2) and (3.3) give rise to external product maps

$$\begin{aligned} \times: K_m(X) \otimes K_n(Y) &\rightarrow K_{m+n}(X \times Y), \\ \times: S_m(X) \otimes K_n(Y) &\rightarrow S_{m+n}(X \times Y). \end{aligned}$$

Remark 3.2. The definitions of the three external products makes use of the (maximal) external tensor product functor in K-theory, which we denote by

$$\boxtimes: K_m(A) \otimes K_n(B) \rightarrow K_{m+n}(A \otimes_{\max} B)$$

in order to distinguish it from our external products. Note that this functor is subject to a sign convention, cf. [HR00b, Remark 4.7.5]¹¹. We will use the sign convention which usually is used in the literature and which has the following compatibility with boundary maps, as seen in [HR00b, Proposition 4.7.6.(b)]¹². If

$$0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$$

is a short exact sequence of C*-algebras and B another C*-algebra, then the sequence

$$0 \rightarrow I \otimes_{\max} B \rightarrow A \otimes_{\max} B \rightarrow (A/I) \otimes_{\max} B \rightarrow 0$$

is also exact and the boundary maps of these two short exact sequences and the external tensor products satisfy the equation

$$\partial(x \boxtimes y) = \partial(x) \boxtimes y$$

for all $x \in K_m(A/I)$ and $y \in K_n(B)$. Using the graded commutativity of the external tensor product we see that the corresponding equation obtained by tensoring the short exact sequence with B from the left and not from the right is only true up to a sign:

$$\partial(y \boxtimes x) = (-1)^n \cdot y \boxtimes \partial(x)$$

Note that this is the sign convention which makes the usual sign heuristics work: exchanging the order of the symbol y of degree n and the symbol ∂ of degree -1 in the last equation gives rise to the sign $(-1)^{n \cdot (-1)}$. \blacklozenge

The *-homomorphisms obtained from (3.1), (3.2) and (3.3) by using the maximal tensor product¹³ fit into a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{L,0}^* X \otimes_{\max} C_L^* Y & \longrightarrow & C_L^* X \otimes_{\max} C_L^* Y & \longrightarrow & C^* X \otimes_{\max} C_L^* Y \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \otimes \text{ev}_1 \\ & & & & & & C^* X \otimes C^* Y \\ & & & & & & \downarrow \\ 0 & \longrightarrow & C_{L,0}^*(X \times Y) & \longrightarrow & C_L^*(X \times Y) & \longrightarrow & C^*(X \times Y) \longrightarrow 0 \end{array}$$

Together with the fact that the external product of K-theory is functorial and compatible with the connecting homomorphisms (cf. Remark 3.2) this gives for any $z \in K_n(Y)$ rise to an external product morphism between long exact sequences:

$$\begin{array}{ccccccc} K_{*+1}(C^* X) & \xrightarrow{\partial} & S_*(X) & \longrightarrow & K_*(X) & \xrightarrow{\text{Ind}} & K_*(C^* X) \\ \downarrow -\times \text{Ind}(z) & & \downarrow -\times z & & \downarrow -\times z & & \downarrow -\times \text{Ind}(z) \\ K_{*+1+n}(C^*(X \times Y)) & \xrightarrow{\partial} & S_{*+n}(X \times Y) & \longrightarrow & K_{*+n}(X \times Y) & \xrightarrow{\text{Ind}} & K_{*+n}(C^*(X \times Y)) \end{array}$$

¹¹This reference treats only the minimal tensor product, but exactly the same is true for the maximal tensor product.

¹²see Footnote 11

¹³The maximal tensor product is needed here to make the upper row exact.

3.2 Equivariant case

Let (H_X, ρ_X) be an X -module as in Section 3.1. We suppose in addition that X is furnished with a proper isometric action of a countable discrete group G . Furthermore, we assume that we have a unitary representation $u_G: G \rightarrow \mathcal{U}(H_X)$ which turns (ρ_X, u_G) into a covariant pair¹⁴. Given this data, we say that (H_X, ρ_X, u_G) is an X - G -module. An X - G -module is called *locally free* if for each finite subgroup $F \subseteq G$ and any F -invariant Borel subset $E \subseteq X$, there is a Hilbert space H_E such that $1_E H_X$ and $\ell^2(F) \otimes H_E$ are isomorphic as F -Hilbert spaces, where $\ell^2(F)$ is endowed with the left-regular representation and H_E is endowed with the trivial representation. An X - G -module is *ample* if it is ample as an X -module and locally free. Ample X - G -modules always exist [WY19, Lemma 4.5.5].

In the following, we let (H_X, ρ_X, u_G) be a fixed ample X - G -module. We also fix an ample Y - H -module (H_Y, ρ_Y, u_H) , where Y is another proper metric space furnished with a proper isometric action of some countable discrete group H .

We get C^* -algebras C_G^*X , $C_{G,L}^*X$ and $C_{G,L,0}^*X$ by considering *equivariant* locally compact operators of finite propagation, respectively suitable families L of them. Similarly as before, their K -theory groups do not depend on the choice of ample X - G -module up to canonical isomorphism [WY19, Theorem 5.2.6, Proposition 7.1.2]. They fit into the short exact sequence

$$0 \rightarrow C_{G,L,0}^*X \rightarrow C_{G,L}^*X \xrightarrow{\text{ev}_1} C_G^*X \rightarrow 0$$

and the induced long exact sequence is then denoted

$$\dots \rightarrow K_{*+1}(C_G^*X) \rightarrow S_*^G(X) \rightarrow K_*^G(X) \xrightarrow{\text{Ind}} K_*(C_G^*X) \rightarrow \dots$$

Writing here $K_*^G(X)$ for $K_*(C_{G,L}^*X)$ is justified by the fact that $K_*(C_{G,L}^*X)$ is naturally isomorphic to the equivariant K -homology of X , see [WY19, Theorem 7.1.2]. If G acts freely on X , then we have

$$K_*(C_{G,L}^*X) \cong K_*^G(X) \cong K_*(G \backslash X),$$

compare Section 5.5.

As in the non-equivariant case we can construct now the external products and get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{G,L,0}^*X \otimes_{\max} C_{H,L}^*Y & \longrightarrow & C_{G,L}^*X \otimes_{\max} C_{H,L}^*Y & \longrightarrow & C_G^*X \otimes_{\max} C_{H,L}^*Y \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \otimes \text{ev}_1 \\ & & & & & & C_G^*X \otimes C_H^*Y \\ & & & & & & \downarrow \\ 0 & \longrightarrow & C_{G \times H, L, 0}^*(X \times Y) & \longrightarrow & C_{G \times H, L}^*(X \times Y) & \longrightarrow & C_{G \times H}^*(X \times Y) \longrightarrow 0 \end{array}$$

which induces for any $z \in K_n^H(Y)$ the external product morphisms between the corresponding long exact sequences:

$$\begin{array}{ccccccc} K_{*+1}(C_G^*X) & \xrightarrow{\partial} & S_*^G(X) & \longrightarrow & K_*^G(X) & \xrightarrow{\text{Ind}} & K_*(C_G^*X) \\ \downarrow -\times \text{Ind}(z) & & \downarrow -\times z & & \downarrow -\times z & & \downarrow -\times \text{Ind}(z) \\ K_{*+1+n}(C_{G \times H}^*(X \times Y)) & \xrightarrow{\partial} & S_{*+n}^{G \times H}(X \times Y) & \longrightarrow & K_{*+n}^{G \times H}(X \times Y) & \xrightarrow{\text{Ind}} & K_{*+n}(C_{G \times H}^*(X \times Y)) \end{array}$$

¹⁴The group G acts on functions on X by $(g \cdot f)(x) = f(g^{-1}x)$. Being a covariant pair means that $u_G(g)\rho_X(f)u_G(g)^* = \rho_X(g \cdot f)$ for all $g \in G$ and $f \in C_0(X)$.

If G acts cocompactly on X , then C_G^*X is Morita equivalent to C_{red}^*G , see [WY19, Prop 5.2.9].

On the universal covers of finite complexes, the action is proper, free and cocompact. In this way we get the diagram considered in Theorem 1.16.

4 Slant products

The goal of this section is to construct the various slant products and prove Theorem 1.11. We start by giving the definition of the stable Higson compactification and corona of a proper metric space Y (which is usually non-compact) and of the coarse co-assembly map from [EM06].

If $\vartheta: Y \rightarrow Z$ is a map into another metric space Z , then one defines for each $r > 0$ the r -variation of ϑ as the function

$$\text{Var}_r \vartheta: Y \rightarrow [0, \infty), \quad x \mapsto \sup\{d(\vartheta(x), \vartheta(y)) \mid y \in Y \text{ with } d(x, y) \leq r\}.$$

The function ϑ is said to have *vanishing variation* if $\text{Var}_r \vartheta$ converges to zero at infinity for all $r > 0$.

Let $\mathcal{K} := \mathcal{K}(\ell^2)$ denote the compact operators on the standard Hilbert space $\ell^2 := \ell^2(\mathbb{N})$. The *stable Higson compactification* of Y , denoted by $\bar{\mathfrak{c}}Y$, is the C^* -algebra of all bounded, continuous functions of vanishing variation $Y \rightarrow \mathcal{K}$. The *stable Higson corona* of Y is the quotient C^* -algebra $\mathfrak{c}Y := \bar{\mathfrak{c}}Y/C_0(Y, \mathcal{K})$.

If Y is unbounded, the stable Higson compactification and corona $\bar{\mathfrak{c}}Y$ and $\mathfrak{c}Y$ contain an isometrically embedded copy of \mathcal{K} as the sub- C^* -algebra of constant functions. Their reduced K-theory is defined as

$$\begin{aligned} \tilde{K}_*(\bar{\mathfrak{c}}Y) &:= K_*(\bar{\mathfrak{c}}Y) / \text{im}(\mathbb{Z} \cong K_*(\mathcal{K}) \rightarrow K_*(\bar{\mathfrak{c}}Y)) \\ \tilde{K}_*(\mathfrak{c}Y) &:= K_*(\mathfrak{c}Y) / \text{im}(\mathbb{Z} \cong K_*(\mathcal{K}) \rightarrow K_*(\mathfrak{c}Y)) \end{aligned}$$

though beware that this definition is only reasonable for unbounded metric spaces [EM06, Rem. 3.9].

There are reduced versions of the above two C^* -algebras whose K-theory behaves better for bounded spaces: Let $\bar{\mathfrak{c}}^{\text{red}}Y$ be the C^* -algebra of all bounded, continuous functions of vanishing variation $Y \rightarrow \mathcal{B}(\ell^2)$ with $f(x) - f(x') \in \mathcal{K}$ for all $x, x' \in Y$. Let $\mathfrak{c}^{\text{red}}Y := \bar{\mathfrak{c}}^{\text{red}}Y/C_0(Y, \mathcal{K})$. We have the isomorphisms

$$K_*(\bar{\mathfrak{c}}^{\text{red}}Y) \cong \tilde{K}_*(\bar{\mathfrak{c}}Y), \quad K_*(\mathfrak{c}^{\text{red}}Y) \cong \tilde{K}_*(\mathfrak{c}Y)$$

if Y is unbounded [EM06, Proposition 5.5] and for bounded Y we simply define the reduced K-theory by these isomorphisms.

The coarse co-assembly map is the connecting homomorphism

$$\mu^*: K_*(\mathfrak{c}^{\text{red}}Y) \rightarrow K^{1-*}(Y)$$

associated to the short exact sequence

$$0 \rightarrow C_0(Y, \mathcal{K}) \rightarrow \bar{\mathfrak{c}}^{\text{red}}Y \rightarrow \mathfrak{c}^{\text{red}}Y \rightarrow 0.$$

In the rest of the paper, we will only use the reduced versions of the stable Higson compactification and corona.

4.1 Construction of the slant products

From now on we assume that X and Y are proper metric spaces, and that Y has bounded geometry (see Definition 4.1.(a)). The representations ρ_X , ρ_Y and $\rho_{X \times Y}$ on the Hilbert spaces H_X , H_Y and $H_{X \times Y} := H_X \otimes H_Y$ shall be exactly as in Section 3.

Definition 4.1 (Bounded geometry). The term *bounded geometry* has in the literature different meanings depending to which kind of object it refers to:

- (a) A metric space Y is said to have *bounded geometry* if there exists $r > 0$ and a subset $\hat{Y} \subset Y$ such that $Y = \bigcup_{\hat{y} \in \hat{Y}} B_r(\hat{y})$ and such that for each $R > 0$ there exists a constant K_R such that for every $y \in Y$ the number of elements $\#(\hat{Y} \cap \bar{B}_R(y))$ is bounded by K_R .

Here $B_r(-)$ denotes the open ball and $\bar{B}_r(-)$ denotes the closed ball of radius r .

- (b) A metric space Y is said to have *continuously bounded geometry* if for every $r > 0$ and $R > 0$ there exists a constant $K_{r,R} > 0$ such that the following conditions hold.

- For every $r > 0$ there is a subset $\hat{Y}_r \subset Y$ such that $Y = \bigcup_{\hat{y} \in \hat{Y}_r} B_r(\hat{y})$ and such that for all $r, R > 0$ and $y \in Y$ the number $\#(\hat{Y}_r \cap \bar{B}_R(y))$ is bounded by $K_{r,R}$.
- For all $\alpha > 0$, we have $K_\alpha := \limsup_{r \rightarrow 0} K_{r, \alpha r} < \infty$.

- (c) A complete Riemannian manifold has *bounded geometry* if it has uniformly positive injectivity radius and the curvature tensor and all its covariant derivatives are uniformly bounded.¹⁵

- (d) A simplicial complex is said to have *bounded geometry* if there is a uniform bound on the number of simplices in the link of each vertex.

Note that this is equivalent to the simplicial complex being uniformly locally finite and finite-dimensional. \blacklozenge

Continuously bounded geometry implies bounded geometry. Bounded geometry of Riemannian manifolds is an even stronger property: It implies continuously bounded geometry of the underlying metric space [Wul19, Lem. 5.2]. Further, if a simplicial complex has bounded geometry, then the underlying metric space will have continuously bounded geometry.

If a metric space Y is uniformly discrete (for instance, a countable discrete group equipped with a proper, left-invariant metric), then it has bounded geometry if and only if it has continuously bounded geometry (and in this case we can take $\hat{Y}_r := Y$ for each $r > 0$).

4.1.1 Slant product for the Roe algebra

We begin by constructing the slant product

$$/: K_p(C^*(X \times Y)) \otimes K_{1-q}(c^{\text{red}}Y) \rightarrow K_{p-q}(C^*X)$$

for X, Y proper metric spaces, and Y of bounded geometry (Definition 4.1.(a)).

¹⁵Note that completeness is actually a redundant requirement here since it is implied by having a uniformly positive injectivity radius.

To construct this slant product we will take the Roe algebra C^*X on the right hand side not to be the one associated to the representation ρ_X , that is $C^*(\rho_X)$, but the one associated to the representation

$$\tilde{\rho}_X := \rho_X \otimes \text{id}_{H_Y \otimes \ell^2}: C_0(X) \rightarrow \mathcal{B}(\tilde{H}_X) \quad (4.1)$$

on $\tilde{H}_X := H_X \otimes H_Y \otimes \ell^2$, that is $C^*(\tilde{\rho}_X)$. Note that the representation $\tilde{\rho}_X$ is ample and non-degenerate, since ρ_X is.

Let $E^*(\tilde{\rho}_X) \subset \mathcal{B}(\tilde{H}_X)$ denote the sub- C^* -algebra generated by all finite propagation operators. Our notation comes from the fact that this C^* -algebra is slightly bigger than the C^* -algebra generated by all pseudolocal operators of finite propagation, which is usually denoted by $D^*(X)$.

Lemma 4.2. *The Roe algebra $C^*(\tilde{\rho}_X)$ is an ideal in $E^*(\tilde{\rho}_X)$.*

Proof. Let $S \in C^*(\tilde{\rho}_X)$ and $T \in E^*(\tilde{\rho}_X)$ have finite propagation. Then $S \circ T$ and $T \circ S$ both have finite propagation, too.

To show local compactness, let $g \in C_0(X)$. We may assume that the support of g is compact and then choose a function $g' \in C_0(X)$ which is constantly equal to 1 on the R -neighborhood of the support of g , where $R > 0$ is the propagation of T . Then the four operators

$$\begin{aligned} \tilde{\rho}_X(g) \circ (S \circ T) &= (\tilde{\rho}_X(g) \circ S) \circ T \\ (T \circ S) \circ \tilde{\rho}_X(g) &= T \circ (S \circ \tilde{\rho}_X(g)) \\ (S \circ T) \circ \tilde{\rho}_X(g) &= (S \circ \tilde{\rho}_X(g')) \circ T \circ \tilde{\rho}_X(g) \\ \tilde{\rho}_X(g) \circ (T \circ S) &= \tilde{\rho}_X(g) \circ T \circ \tilde{\rho}_X(g') \circ S \end{aligned}$$

are clearly compact, because S is locally compact. Density arguments finish the proof. \square

Observe that the tensor product of the representation ρ_Y and the canonical representation of \mathcal{K} on ℓ^2 is a non-degenerate representation of $C_0(Y, \mathcal{K})$ on $H_Y \otimes \ell^2$. Thus it extends uniquely to a strictly continuous representation

$$\bar{\rho}_Y: \mathcal{M}(C_0(Y, \mathcal{K})) \rightarrow \mathcal{B}(H_Y \otimes \ell^2) \quad (4.2)$$

of the multiplier algebra and subsequent tensoring with the identity on H_X gives us a representation

$$\tilde{\rho}_Y := \text{id}_{H_X} \otimes \bar{\rho}_Y: \mathcal{M}(C_0(Y, \mathcal{K})) \rightarrow \mathcal{B}(\tilde{H}_X)$$

whose image clearly commutes with the image of the representation $\tilde{\rho}_X$. Since $C_0(Y, \mathcal{K})$ is an essential ideal in $C_b(Y, \mathcal{K})$, the latter embeds canonically into $\mathcal{M}(C_0(Y, \mathcal{K}))$ and we denote the restrictions of $\bar{\rho}_Y$ and $\tilde{\rho}_Y$ to $C_b(Y, \mathcal{K})$ and $C_0(Y, \mathcal{K})$ by the same letters.

Lemma 4.3. *The images of the two representations*

$$\begin{aligned} \tau: C^*(\rho_{X \times Y}) &\rightarrow \mathcal{B}(\tilde{H}_X) \text{ given by } S \mapsto S \otimes \text{id}_{\ell^2}, \\ \tilde{\rho}_Y: \mathcal{M}(C_0(Y, \mathcal{K})) &\rightarrow \mathcal{B}(\tilde{H}_X) \text{ defined above,} \end{aligned}$$

are contained in $E^*(\tilde{\rho}_X)$.

Proof. The propagation of each operator in the image of $\tilde{\rho}_Y$ is clearly zero. For τ , the claim follows from the following important lemma. \square

Lemma 4.4. *If $S \in \mathcal{B}(H_{X \times Y})$ has finite propagation bounded by $R > 0$ with respect to $\rho_{X \times Y}$, then the operator $S \otimes \text{id}_{\ell^2} \in \mathcal{B}(\tilde{H}_X)$ has finite propagation bounded by R with respect to $\tilde{\rho}_X$.*

Proof. Let $g, h \in C_0(X)$ be functions whose supports are further apart than R . Then $\tilde{\rho}_X(g) \circ (S \otimes \text{id}_{\ell^2}) \circ \tilde{\rho}_X(h)$ can be written as a strong limit of operators of the form $(\rho_{X \times Y}(g \otimes \varphi) \circ S \circ \rho_{X \times Y}(h \otimes \varphi)) \otimes \text{id}_{\ell^2}$ with $\varphi \in C_0(Y)$, but the latter are all zero, because the supports of $g \otimes \varphi, h \otimes \varphi \in C_0(X \times Y)$ are further apart than R . \square

Lemma 4.5. *The image of the representation τ commutes up to $C^*(\tilde{\rho}_X)$ with the image of $\tilde{\tau}^{\text{red}} Y$ under the representation $\tilde{\rho}_Y$.*

Hence, by the universal property of the maximal tensor product, we get an induced $*$ -homomorphism

$$\Phi: C^*(\rho_{X \times Y}) \otimes_{\max} \tilde{\tau}^{\text{red}} Y \rightarrow E^*(\tilde{\rho}_X)/C^*(\tilde{\rho}_X) \quad (4.3)$$

given by $S \otimes f \mapsto [\tau(S) \circ \tilde{\rho}_Y(f)]$.

Proof. Let $S \in C^*(\rho_{X \times Y})$ and $f \in \tilde{\tau}^{\text{red}} Y$. We may assume that S has finite propagation with respect to $\rho_{X \times Y}$ and hence $\tau(S)$ also has finite propagation with respect to $\tilde{\rho}_X$ by the Lemma 4.4. As $\tilde{\rho}_Y(f)$ also has finite propagation (namely zero) with respect to $\tilde{\rho}_X$, so does the commutator $[\tau(S), \tilde{\rho}_Y(f)]$.

Therefore it remains to show that the commutator is also locally compact with respect to $\tilde{\rho}_X$. Here we need the bounded geometry of Y .

Choose a subset $\hat{Y} \subset Y$ as in Definition 4.1.(a), i. e. with $Y = \bigcup_{\hat{y} \in \hat{Y}} B_r(\hat{y})$ and such that for each $R' > 0$ the number $\#(\hat{Y} \cap \overline{B}_{R'}(y))$ is uniformly bounded in $y \in Y$ by some constant $K_{R'} > 0$. In this proof, the relevant value for R' will be $R' = R + 2r$, where R is the propagation of the operator S .

By thinning out the open cover $\{B_r(\hat{y})\}_{\hat{y} \in \hat{Y}}$, we obtain a decomposition of Y into a family $\{Z_{\hat{y}}\}_{\hat{y} \in \hat{Y}}$ of pairwise disjoint Borel subsets such that $Z_{\hat{y}} \subseteq B_r(\hat{y})$ for all $\hat{y} \in \hat{Y}$. The representation $\rho_Y: C_0(Y) \rightarrow \mathcal{B}(H_Y)$ extends uniquely to the bounded Borel functions on Y subject to the condition that pointwise converging uniformly bounded sequences of functions are taken to strongly converging sequences of operators. For a Borel subset $Z \subseteq Y$, let $1_Z \in \mathcal{B}(H_Y)$ denote the projection corresponding to the characteristic function of Z . Now consider the strongly convergent series

$$\hat{f} := \sum_{\hat{y} \in \hat{Y}} 1_{Z_{\hat{y}}} \otimes f(\hat{y}) \in \mathcal{B}(H_Y \otimes \ell^2).$$

The proof will be completed by showing first that the operators $(\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}(f)) \circ \tau(S)$ and $\tau(S) \circ (\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}(f))$ are locally compact and second that the commutator $[\tau(S), \text{id}_{H_X} \otimes \hat{f}]$ is locally compact.

Let $h_1 \leq h_2 \leq \dots$ be a sequence of compactly supported functions $Y \rightarrow [0, 1]$ such that the compact subsets $h_n^{-1}\{1\}$ exhaust Y as $n \rightarrow \infty$. Furthermore, let $P_n \in \mathcal{K}$ denote the projection onto the span of the first n basis vectors $\delta_1, \dots, \delta_n \in \ell^2$. It is clear from the construction of \hat{f} together with the vanishing variation of f that

$$(\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}_Y(f)) = \lim_{n \rightarrow \infty} (\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}_Y(f)) \circ \tilde{\rho}_Y(h_n \otimes P_n)$$

with convergence in norm. Using this, we find for every $g \in C_0(X)$ the equation

$$\begin{aligned} \tilde{\rho}_X(g) \circ (\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}_Y(f)) \circ \tau(S) &= \\ &= \lim_{n \rightarrow \infty} (\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}_Y(f)) \circ ((\rho_{X \times Y}(g \otimes h_n) \circ S) \otimes P_n) \end{aligned}$$

where the right hand side is a norm limit of compact operators, hence itself compact.

If $g \in C_0(X)$ has compact support, then as in the proof of Lemma 4.2 we can choose $g' : X \rightarrow [0, 1]$ of compact support which is equal to 1 on the R -neighborhood of the support of g and one similarly obtains compactness the operator

$$(\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}_Y(f)) \circ \tau(S) \circ \tilde{\rho}_X(g) = (\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}_Y(f)) \circ \tilde{\rho}_X(g') \circ \tau(S) \circ \tilde{\rho}_X(g).$$

We have thus shown that $(\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}_Y(f)) \circ \tau(S)$ is locally compact and analogously we obtain local compactness of $\tau(S) \circ (\text{id}_{H_X} \otimes \hat{f} - \tilde{\rho}_Y(f))$, too.

It remains to show local compactness of the commutator

$$\begin{aligned} [\tau(S), \hat{f}] &= \tau(S) \circ \sum_{\hat{z} \in \hat{Y}} \text{id}_{H_X} \otimes 1_{Z_{\hat{z}}} \otimes f(\hat{z}) - \sum_{\hat{y} \in \hat{Y}} (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}} \otimes f(\hat{y})) \circ \tau(S) \\ &= \sum_{\hat{y}, \hat{z} \in \hat{Y}} \left((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}} \circ S \circ (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}}) \right) \otimes (f(\hat{z}) - f(\hat{y})) \\ &= \sum_{\substack{\hat{y}, \hat{z} \in \hat{Y} \\ d(\hat{y}, \hat{z}) \leq R+2r}} \left((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}} \circ S \circ (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}}) \right) \otimes (f(\hat{z}) - f(\hat{y})) \end{aligned}$$

where the sums converge a priori in the strong operator topology. Let us first show that the last sum converges even in norm. Because of the vanishing variation of f there is a finite subset $L \subset \hat{Y}$ such that $\|f(\hat{z}) - f(\hat{y})\| < \varepsilon$ whenever $\hat{y}, \hat{z} \in \hat{Y} \setminus L$ satisfy $d(\hat{y}, \hat{z}) \leq R+2r$. For arbitrary $v \in \tilde{H}_X$ the vectors $v_{\hat{y}} := (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}} \otimes \text{id}_{\ell^2})v$ for $\hat{y} \in \hat{Y}$ are pairwise orthogonal and $v = \sum_{\hat{y} \in \hat{Y}} v_{\hat{y}}$, hence $\|v\|^2 = \sum_{\hat{y} \in \hat{Y}} \|v_{\hat{y}}\|^2$. Then the calculation

$$\begin{aligned} &\left\| \sum_{\substack{\hat{y}, \hat{z} \in \hat{Y} \setminus L \\ d(\hat{y}, \hat{z}) \leq R+2r}} \left((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}} \circ S \circ (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}}) \right) \otimes (f(\hat{z}) - f(\hat{y})) v \right\|^2 = \\ &= \sum_{\hat{z} \in \hat{Y} \setminus L} \left\| \sum_{\substack{\hat{y} \in \hat{Y} \setminus L \\ d(\hat{y}, \hat{z}) \leq R+2r}} \left((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}} \circ S \circ (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}}) \right) \otimes (f(\hat{z}) - f(\hat{y})) v \right\|^2 \\ &\leq \sum_{\hat{y} \in \hat{Y} \setminus L} (K_{R+2r} \cdot \|S\| \cdot \varepsilon \cdot \|v_{\hat{y}}\|)^2 \\ &\leq K_{R+2r}^2 \cdot \|S\|^2 \cdot \varepsilon^2 \cdot \|v\|^2 \end{aligned}$$

shows the claimed norm convergence.

Now, given $g \in C_0(X)$ and choosing for each $\hat{z} \in \hat{Y}$ a compactly supported function $h_{\hat{z}}$ which is constantly 1 on $B_r(\hat{z})$, we find that

$$\begin{aligned} \tilde{\rho}_X(g) \circ [\tau(S), \hat{f}] &= \\ &= \sum_{\substack{\hat{y}, \hat{z} \in \hat{Y} \\ d(\hat{y}, \hat{z}) \leq R+2r}} \left((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}} \circ \underbrace{\rho_{X \times Y}(g \otimes h_{\hat{z}})}_{\in \mathcal{K}(H_{X \times Y})} \circ S \circ (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}}) \right) \otimes \left(\underbrace{f(\hat{z}) - f(\hat{y})}_{\in \mathcal{K}} \right) \end{aligned}$$

is a norm convergent sum of compact operators, hence itself compact. Analogously, $[\tau(S), \hat{f}] \circ \tilde{\rho}_X(g)$ is compact, and hence the commutator $[\tau(S), \hat{f}]$ is locally compact. \square

Lemma 4.6. *The $*$ -homomorphism Φ from (4.3) factors through the tensor product $C^*(\rho_{X \times Y}) \otimes_{\max} \mathfrak{c}^{\text{red}} Y$. In other words, it defines a $*$ -homomorphism*

$$\Psi : C^*(\rho_{X \times Y}) \otimes_{\max} \mathfrak{c}^{\text{red}} Y \rightarrow E^*(\tilde{\rho}_X) / C^*(\tilde{\rho}_X). \quad (4.4)$$

Proof. Due to exactness of the maximal tensor product, the claim is equivalent to Φ vanishing on $C^*(\rho_{X \times Y}) \otimes C_0(Y, \mathcal{K}) = C^*(\rho_{X \times Y}) \otimes C_0(Y) \otimes \mathcal{K}$. Therefore, given operators $S \in C^*(\rho_{X \times Y})$ and $f \otimes T \in C_0(Y) \otimes \mathcal{K}$ we have to show that $\tau(S) \circ \tilde{\rho}_Y(f \otimes T) \in C^*(\tilde{\rho}_X)$.

The finite propagation part of this statement is proven exactly as the one in the preceding lemma. For local compactness we use the formula

$$\tau(S) \circ \tilde{\rho}_Y(f \otimes T) \circ \tilde{\rho}_X(g) = (S \circ \rho_{X \times Y}(g \otimes f)) \otimes T \in \mathcal{K}(H_{X \times Y}) \otimes \mathcal{K}(\ell^2)$$

and a similar one for $\tilde{\rho}_X(g) \circ \tau(S) \circ \tilde{\rho}_Y(f \otimes T)$, involving the same function g' as in the previous lemma. \square

Definition 4.7. The slant product between K-theory of the Roe algebra and the K-theory of the reduced stable Higson corona is now defined as $(-1)^p$ times the composition

$$\begin{aligned} K_p(C^*(X \times Y)) \otimes K_{1-q}(\mathbf{c}^{\text{red}}Y) &= K_p(C^*(\rho_{X \times Y})) \otimes K_{1-q}(\mathbf{c}^{\text{red}}Y) \\ &\rightarrow K_{p+1-q}(C^*(\rho_{X \times Y}) \otimes_{\max} \mathbf{c}^{\text{red}}Y) \\ &\xrightarrow{\Psi_*} K_{p+1-q}(E^*(\tilde{\rho}_X)/C^*(\tilde{\rho}_X)) \\ &\xrightarrow{\partial} K_{p-q}(C^*(\tilde{\rho}_X)) \\ &= K_{p-q}(C^*X), \end{aligned} \tag{4.5}$$

where the first arrow is the external product on K-theory, and the third arrow the boundary operator in the corresponding long exact sequence. \blacklozenge

4.1.2 Slant products for the localization algebras

To construct the analogous slant products for the localization algebras we use the same approach as for the construction of the slant product on the Roe algebra in the previous section.

We define $E_L^*(\tilde{\rho}_X)$ as the C^* -subalgebra of $C_b([1, \infty), E^*(\tilde{\rho}_X))$ generated by the bounded and uniformly continuous functions $S: [1, \infty) \rightarrow E^*(\tilde{\rho}_X)$ such that the propagation of $S(t)$ is finite for all $t \geq 1$ and tends to zero as $t \rightarrow \infty$. Similarly we define $E_{L,0}^*(\tilde{\rho}_X)$ as the ideal in $E_L^*(\tilde{\rho}_X)$ consisting of all maps that vanish at 1. Note that $C_L^*(\tilde{\rho}_X)$ is an ideal in $E_L^*(\tilde{\rho}_X)$ and $C_{L,0}^*(\tilde{\rho}_X)$ is even an ideal in all of the three $E_L^*(\tilde{\rho}_X)$, $E_{L,0}^*(\tilde{\rho}_X)$ and of course $C_L^*(\tilde{\rho}_X)$.

Lemma 4.8. *The following analogues of Lemmas 4.3, 4.5 and 4.6 hold true:*

- (i) *The images of the two isometric $*$ -homomorphisms*

$$\tau_L: C_L^*(\rho_{X \times Y}) \rightarrow C_b([1, \infty), \mathcal{B}(\tilde{H}_X)),$$

which is obtained by applying the functor $C_b([1, \infty), -)$ to τ , and

$$\tilde{\rho}_{Y,L}: \mathcal{M}(C_0(Y, \mathcal{K})) \xrightarrow{\tilde{\rho}_Y} \mathcal{B}(\tilde{H}_X) \xrightarrow[\text{as constant functions}]{\text{inclusion}} C_b([1, \infty), \mathcal{B}(\tilde{H}_X)),$$

are contained in $E_L^(\tilde{\rho}_X)$.*

- (ii) *The image of τ_L commutes up to $C_L^*(\tilde{\rho}_X)$ with the image of $\bar{\mathbf{c}}^{\text{red}}Y$ under $\tilde{\rho}_{Y,L}$ and the image of $C_{L,0}^*(\rho_{X \times Y})$ under τ_L commutes up to $C_{L,0}^*(\tilde{\rho}_X)$ with the image of $\bar{\mathbf{c}}^{\text{red}}Y$ under $\tilde{\rho}_{Y,L}$. Hence they induce $*$ -homomorphisms*

$$\begin{aligned} \Phi_L: C_L^*(\rho_{X \times Y}) \otimes_{\max} \bar{\mathbf{c}}^{\text{red}}Y &\rightarrow E_L^*(\tilde{\rho}_X)/C_L^*(\tilde{\rho}_X) \\ \Phi_{L,0}: C_{L,0}^*(\rho_{X \times Y}) \otimes_{\max} \bar{\mathbf{c}}^{\text{red}}Y &\rightarrow E_{L,0}^*(\tilde{\rho}_X)/C_{L,0}^*(\tilde{\rho}_X) \end{aligned}$$

given by $S \otimes f \mapsto [\tau_L(S) \circ \tilde{\rho}_{Y,L}(f)]$ and the image of $\Phi_{L,0}$ is even contained in $E_{L,0}^*(\tilde{\rho}_X)/C_{L,0}^*(\tilde{\rho}_X)$.

- (iii) The $*$ -homomorphisms Φ_L and $\Phi_{L,0}$ factor through $C_L^*(\rho_{X \times Y}) \otimes_{\max} \mathfrak{c}^{\text{red}}Y$ and $C_{L,0}^*(\rho_{X \times Y}) \otimes_{\max} \mathfrak{c}^{\text{red}}Y$, respectively. That is, they define $*$ -homomorphisms

$$\begin{aligned}\Psi_L &: C_L^*(\rho_{X \times Y}) \otimes_{\max} \mathfrak{c}^{\text{red}}Y \rightarrow E_L^*(\tilde{\rho}_X)/C_L^*(\tilde{\rho}_X), \\ \Psi_{L,0} &: C_{L,0}^*(\rho_{X \times Y}) \otimes_{\max} \mathfrak{c}^{\text{red}}Y \rightarrow E_{L,0}^*(\tilde{\rho}_X)/C_{L,0}^*(\tilde{\rho}_X).\end{aligned}$$

Proof. The estimates on the propagation in these lemmas rely on Lemma 4.4 and on the fact that the propagation of the composition of operators is at most the sum of the propagations of the summands, and so we are still fine in our situation here. And due to our definition of $C_L^*(-)$ we have to check local compactness in the proofs of the analogous versions of Lemmas 4.5 and 4.6 only point-wise in time, i.e. for fixed $t \in [1, \infty)$, and hence we can directly use the corresponding arguments from the proofs of the Lemmas 4.5 and 4.6. \square

Definition 4.9. The slant products

$$\begin{aligned}K_p(X \times Y) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y) &\rightarrow K_{p-q}(X) \\ S_p(X \times Y) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y) &\rightarrow S_{p-q}(X)\end{aligned}$$

are defined as $(-1)^p$ times compositions analogous to that of Definition 4.7 but using the maps Ψ_L and $\Psi_{L,0}$, respectively, instead of Ψ . \blacklozenge

4.2 Compatibility with the Higson–Roe sequence

In this section we prove the following compatibility of the slant products with the Higson–Roe sequence.

Theorem 4.10. *The diagram*

$$\begin{array}{ccccccc}S_p(X \times Y) & \longrightarrow & K_p(X \times Y) & \longrightarrow & K_p(C^*(X \times Y)) & \xrightarrow{\partial} & S_{p-1}(X \times Y) \\ \downarrow / \theta & & \downarrow / \theta & & \downarrow / \theta & & \downarrow / \theta \\ S_{p-q}(X) & \longrightarrow & K_{p-q}(X) & \longrightarrow & K_{p-q}(C^*X) & \xrightarrow{\partial} & S_{p-1-q}(X)\end{array}$$

commutes for every $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Y)$.

Proof. Consider the diagram

$$\begin{array}{ccccccc}K_p(C_{L,0}^*(X \times Y)) & \longrightarrow & K_p(C_L^*(X \times Y)) & \longrightarrow & K_p(C^*(X \times Y)) & \xrightarrow{\partial} & K_{p-1}(C_{L,0}^*(X \times Y)) \\ \downarrow \boxtimes \theta & & \downarrow \boxtimes \theta & & \downarrow \boxtimes \theta & & \downarrow \boxtimes \theta \\ K_{p+1-q}(C_{L,0}^*(X \times Y) \otimes_{\max} \mathfrak{c}^{\text{red}}Y) & \longrightarrow & K_{p+1-q}(C_L^*(X \times Y) \otimes_{\max} \mathfrak{c}^{\text{red}}Y) & \longrightarrow & K_{p+1-q}(C^*(X \times Y) \otimes_{\max} \mathfrak{c}^{\text{red}}Y) & \xrightarrow{\partial} & K_{p-q}(C_{L,0}^*(X \times Y) \otimes_{\max} \mathfrak{c}^{\text{red}}Y) \\ \downarrow (\Psi_{L,0})_* & & \downarrow (\Psi_L)_* & & \downarrow \Psi_* & & \downarrow (\Psi_{L,0})_* \\ K_{p+1-q}\left(\frac{E_{L,0}^*X}{C_{L,0}^*X}\right) & \longrightarrow & K_{p+1-q}\left(\frac{E_L^*X}{C_L^*X}\right) & \longrightarrow & K_{p+1-q}\left(\frac{E^*X}{C^*X}\right) & \xrightarrow{\partial} & K_{p-q}\left(\frac{E_{L,0}^*X}{C_{L,0}^*X}\right) \\ \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ K_{p-q}(C_{L,0}^*X) & \longrightarrow & K_{p-q}(C_L^*X) & \longrightarrow & K_{p-q}(C^*X) & \xrightarrow{\partial} & K_{p-1-q}(C_{L,0}^*X)\end{array}$$

whose rows are the long exact sequences in K-theory induced by the obvious short exact sequences of C*-algebras and whose vertical compositions are the slant products defined in the previous section up to the signs $(-1)^p$ for the left three columns and $(-1)^{p-1}$ for the right column.

Commutativity of the upper three squares is a well known property of the external tensor product in K-theory, i. e. functoriality for the left two squares and the sign convention of Remark 3.2 for the square to the right.

Commutativity of the three squares in the middle row and the left two squares in the bottom row are due to naturality of the long exact sequence in K-theory, considering that the three *-homomorphisms $\Psi_{L,0}$, Ψ_L and Ψ together comprise a morphism of short exact sequences and considering that we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_{L,0}^*(X) & \longrightarrow & C_L^*(X) & \longrightarrow & C^*(X) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E_{L,0}^*(X) & \longrightarrow & E_L^*(X) & \longrightarrow & E^*(X) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E_{L,0}^*(X)/C_{L,0}^*(X) & \longrightarrow & E_L^*(X)/C_L^*(X) & \longrightarrow & E^*(X)/C^*(X) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

It is an abstract fact that the outer boundary maps associated to such a grid of exact sequences commute up to multiplication with -1 . This proves commutativity of the bottom right square up to -1 , and this extra sign is exactly the one needed to match the difference of the signs implemented in the slant products. \square

4.3 The slant product on K-homology

In this section we are going to compare the slant product for the localization algebra C_L^* with the usual slant product between K-homology and K-theory. There are several ways to define the latter, so let us specify that the definition which we want to work with is the one obtained from E -theory.

Recall that E -theory is a bivariant K-theory for C*-algebras, i. e. it has properties analogous to those of KK-theory, and it even agrees with KK-theory on nuclear and separable C*-algebras. One recovers the K-homology and K-theory groups of a locally compact Hausdorff space X as the special cases $K_*(X) \cong E_*(C_0(X), \mathbb{C})$ and $K^*(X) \cong E_{-*}(\mathbb{C}, C_0(X))$. The reason why we prefer E -theory over KK-theory is that the localization algebras are closely related to asymptotic morphisms as in the definition of E -theory. Even more, the isomorphism between the K-theory of the localization algebra and K-homology which is given in [QR10, Corollary 4.2 and Proposition 4.3] (see Proposition 4.14 below) is in fact given as an isomorphism

$$\Delta: K_*(C_L^*X) \xrightarrow{\cong} E_*(C_0(X), \mathbb{C}).$$

Our standard references for E -theory are the papers [GHT00; GH04]. Note that they actually only consider the E -theory groups $E_0(-, -)$ in degree zero, but the higher E -theory groups are obtained from them in the usual way, i. e. by tensoring

suitably with $C_0(\mathbb{R}^n)$ (or with the \mathbb{Z}_2 -graded Clifford algebras $\mathbb{C}\ell_n$), just as it is done in KK-theory. Therefore, also the same sign heuristics as in KK-theory apply.

There is just one subtle difference between E -theory and KK-theory which one has to get right to make the sign heuristics work: In KK-theory, the Kasparov product of $x \in \text{KK}_m(A, B)$ and $y \in \text{KK}_n(B, C)$ is usually written as $x \otimes_B y \in \text{KK}_{m+n}(A, C)$, whereas in E -theory, the composition product of $x \in E_m(A, B)$ and $y \in E_n(B, C)$ is usually written like a composition of functions as $y \circ x \in E_{m+n}(A, C)$. Comparing these two notations one realizes that the order of x and y is exchanged and hence it should be expected that they agree only up to a sign $(-1)^{mn}$. This is indeed the case, and it is due to the fact that in taking the products one has to choose identifications $C_0(\mathbb{R}^m) \otimes C_0(\mathbb{R}^n) \cong C_0(\mathbb{R}^{m+n})$, and these identifications have to be chosen differently for E -theory than for KK-theory to make the sign heuristics work.

Maybe the best way to visualize this is the commutative diagram

$$\begin{array}{ccc} \text{KK}_m(A, B) \otimes \text{KK}_n(B, C) & \xrightarrow{\otimes_B} & \text{KK}_{m+n}(A, C) \\ \downarrow (-1)^{mn} \cdot \text{flip} & & \downarrow \\ E_n(B, C) \otimes E_m(A, B) & \xrightarrow{\circ} & E_{m+n}(A, C) \end{array}$$

where the left vertical arrow not only maps KK-theory to E -theory and exchanges the two factors, but also multiplies by $(-1)^{mn}$.

Note that this difference sticks out in the special case $m+n=0$, $A=C=\mathbb{C}$ and $B=C_0(X)$: For $x \in K_n(X)$ and $y \in K^n(X)$ the E -theoretic pairing is $\langle x, y \rangle := x \circ y \in E_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ and the KK-theoretic pairing is $\langle y, x \rangle := y \otimes_{C_0(X)} x \in \text{KK}_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ and these two pairings differ by $(-1)^{-n^2} = (-1)^n$,

$$\langle y, x \rangle = (-1)^n \langle x, y \rangle,$$

and thanks to the choice of the order of x, y in these notations of the pairing, the formula is again compatible with the sign heuristics. We will always use the E -theoretic version of the pairing in this paper.

Definition 4.11. The slant product

$$\begin{aligned} K_p(X \times Y) \otimes K^q(Y) &\rightarrow K_{p-q}(X) \\ x \otimes \theta &\mapsto x/\theta \end{aligned}$$

is defined as

$$\begin{aligned} E_p(C_0(X) \otimes C_0(Y), \mathbb{C}) \otimes E_{-q}(\mathbb{C}, C_0(Y)) &\rightarrow E_{p-q}(C_0(X), \mathbb{C}) \\ x \otimes \theta &\mapsto x \circ (\text{id}_{C_0(X)} \boxtimes \theta). \end{aligned}$$

Note that for Y a one-point space and $p=q$ one recovers the pairing as a special case of the slant product: $\langle x, \theta \rangle = x/\theta$. \blacklozenge

The two properties of the following lemma are in accordance with the sign heuristics and their proofs are straightforward.

Lemma 4.12. For $x \in K_m(X)$, $y \in K_p(Y \times Z)$ and $\theta \in K^q(Z)$ we have

$$(x \times y)/\theta = x \times (y/\theta)$$

and for $x \in K_p(X \times Y \times Z)$, $\eta \in K^q(Y)$ and $\theta \in K^r(Z)$ we have

$$x/(\eta \times \theta) = (-1)^{qr} \cdot (x/\theta)/\eta.$$

The remainder of this section is devoted to proving the following comparison theorem between the E -theoretic slant product and the slant product which we had defined for the localization algebra.

Theorem 4.13. *Let Y have continuously bounded geometry. Then the slant product on the localization algebra from Definition 4.9 and the E -theoretic slant product are related via the co-assembly map $\mu^*: K_{1-q}(\mathfrak{c}^{\text{red}}Y) \rightarrow K^q(Y)$ and the isomorphisms $\Delta: K_*(C_L^*(-)) \xrightarrow{\cong} K_*(-)$ by the commutative diagram*

$$\begin{array}{ccc} K_p(C_L^*(X \times Y)) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y) & \xrightarrow{\quad / \quad} & K_{p-q}(C_L^*X) \\ \Delta \otimes \mu^* \downarrow & & \Delta \downarrow \cong \\ K_p(X \times Y) \otimes K^q(Y) & \xrightarrow{\quad / \quad} & K_{p-q}(X) \end{array}$$

Let us begin by recalling the isomorphisms Δ between the K -theory of the localization algebra and the K -homology groups.

Proposition 4.14 (compare [QR10, Corollary 4.2 and Proposition 4.3]). *Let X be a proper metric space and (H_X, ρ_X) an X -module. Then*

$$\begin{aligned} \delta(\rho_X): C_L^*(\rho_X) \otimes C_0(X) &\rightarrow C_b([1, \infty), \mathcal{K}(H_X)) / C_0([1, \infty), \mathcal{K}(H_X)) \\ (T_t)_{t \in [1, \infty)} \otimes f &\mapsto [(T_t \circ \rho_X(f))_{t \in [1, \infty)}] \end{aligned}$$

defines an asymptotic morphism and hence a canonical element of $E_0(C_L^*(\rho_X) \otimes C_0(X), \mathbb{C})$. The isomorphism between the K -theory of the localization algebra constructed with an ample X -module (H_X, ρ_X) and K -homology is given by the E -theoretic composition product

$$\begin{aligned} \Delta(\rho_X): K_m(C_L^*(\rho_X)) &\xrightarrow{\cong} E_m(C_0(X), \mathbb{C}) \cong K_m(X) \\ x &\mapsto \delta(\rho_X) \circ (x \boxtimes \text{id}_{C_0(X)}). \end{aligned}$$

For later purposes we note that $\delta(\rho_X)$ obviously factorizes through the quotient $(C_L^*(\rho_X) / C_0([1, \infty), C^*(\rho_X))) \otimes C_0(X)$.

The first part of this proposition is an easy corollary of the following well-known technical lemma, which we will also need to exploit several further times below.

Lemma 4.15 (compare [QR10, Proposition 4.1.], [WY19, Lemma 6.1.2]). *There exists a constant $C > 0$ such that the following holds. Let (H_X, ρ_X) be an X -module and $\vartheta: X \rightarrow \mathbb{C}$ a bounded Borel function. Then for every operator $T \in \mathcal{B}(K)$ the estimate*

$$\|[T, \rho_X(\vartheta)]\| \leq C \cdot \|T\| \cdot \|\text{Var}_{\text{prop}(T)} \vartheta\|$$

holds.

Note that ϑ is uniformly continuous iff $\|\text{Var}_r \vartheta\| \rightarrow 0$ as $r \rightarrow 0$.

Lemma 4.16. *Let $(T_t)_{t \in [1, \infty)} \in C_L^*(\rho_{X \times Y})$ and $f \in C_0(Y, \mathcal{K})$. Then*

$$\lim_{t \rightarrow \infty} [\tau(T_t), \tilde{\rho}_Y(f)] = 0. \quad (4.6)$$

Proof. Since $C_0(Y, \mathcal{K}) = C_0(Y) \otimes \mathcal{K}$, we assume by an approximation argument without loss of generality that $f = \tilde{f} \otimes K$, where $\tilde{f} \in C_0(Y)$ and $K \in \mathcal{K}$. Moreover, we can assume that $\lim_{t \rightarrow \infty} \text{prop}(T_t) = 0$. Then Lemma 4.15 implies that there exists $C \geq 0$ such that

$$\|[T_t, \text{id}_{H_X} \otimes \rho_Y(\tilde{f})]\| \leq C \|T_t\| \|\text{Var}_{\text{prop}(T_t)} 1 \otimes \tilde{f}\|.$$

We have $\|\text{Var}_r 1 \otimes \tilde{f}\| \rightarrow 0$ as $r \rightarrow 0$ because $1 \otimes \tilde{f}: X \times Y \rightarrow \mathbb{C}$ is uniformly continuous. Hence $[T_t, \text{id}_{H_X} \otimes \rho_Y(\tilde{f})] \rightarrow 0$ as $t \rightarrow \infty$. We conclude that

$$\begin{aligned} \lim_{t \rightarrow \infty} [\tau(T_t), \tilde{\rho}_Y(f)] &= \lim_{t \rightarrow \infty} [T_t \otimes \text{id}_{\ell^2}, \text{id}_{H_X} \otimes \rho_Y(\tilde{f}) \otimes K] \\ &= \lim_{t \rightarrow \infty} [T_t, \text{id}_{H_X} \otimes \rho_Y(\tilde{f})] \otimes K = 0. \end{aligned} \quad \square$$

Lemma 4.15 only applies to scalar-valued functions. Therefore the tensor product decomposition $C_0(Y, \mathcal{K}) = C_0(Y) \otimes \mathcal{K}$ was a crucial aspect in the argument above. In particular, this argument does not prove (4.6) for functions in the stable Higson compactification, although it would work for the usual (unstable) Higson compactification. However, under the assumption of continuously bounded geometry, a more difficult argument yields (4.6) also for functions in the stable Higson compactification, which is the content of the following proposition.

Proposition 4.17. *Suppose that Y has continuously bounded geometry. Let $(T_t)_{t \in [1, \infty)} \in C_L^*(\rho_{X \times Y})$ and $f: Y \rightarrow \mathcal{B}(\ell^2)$ bounded and uniformly continuous. Then*

$$\lim_{t \rightarrow \infty} [\tau(T_t), \tilde{\rho}_Y(f)] = 0. \quad (4.7)$$

Proof. The argument is similar as in the proof of [Wul19, Lemma 5.6] and Lemma 4.5 above. First, we assume as usual that $\lim_{t \rightarrow \infty} \text{prop}(T_t) = 0$. Now let $\varepsilon > 0$. Choose $\delta > 0$ such that $\|f(y) - f(y')\| < \varepsilon$ if $d(y, y') < \delta$. Let $r := \delta/4$. Let $\hat{Y}_r \subseteq Y$ be a subset witnessing our conditions for continuously bounded geometry. After perhaps making $\delta = 4r$ smaller, we can ensure that $K_{r, 4r} \leq K_4 + 1 < \infty$. By thinning out the cover $(B_r(\hat{y}))_{\hat{y} \in \hat{Y}_r}$, we obtain a pairwise disjoint cover $(Z_{\hat{y}})_{\hat{y} \in \hat{Y}_r}$ consisting of Borel sets such that $Z_{\hat{y}} \subseteq B_r(\hat{y})$ for all $\hat{y} \in \hat{Y}_r$. The representation $\rho_Y: C_0(Y) \rightarrow \mathcal{B}(H_Y)$ extends uniquely to the bounded Borel functions on Y subject to the condition that pointwise converging uniformly bounded sequences of functions are taken to strongly converging sequences of operators. For a Borel subset $Z \subseteq Y$, let $1_Z \in \mathcal{B}(H_Y)$ denote the operator corresponding to the characteristic function of Z . Now consider the strongly convergent series $\hat{f} := \sum_{\hat{y} \in \hat{Y}_r} 1_{Z_{\hat{y}}} \otimes f(\hat{y}) \in \mathcal{B}(H_Y \otimes \ell^2)$. Then $\|\tilde{\rho}(f) - \hat{f}\| \leq \varepsilon$. We can write the commutator of $\tau(T_t) = T_t \otimes \text{id}_{\ell^2}$ and $\text{id}_{H_X} \otimes \hat{f}$ as follows.

$$\begin{aligned} [T_t \otimes \text{id}_{\ell^2}, \text{id}_{H_X} \otimes \hat{f}] &= \sum_{\hat{y}, \hat{z} \in \hat{Y}_r} \left((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}}) T_t (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}}) \right) \otimes (f(\hat{y}) - f(\hat{z})) \\ &= \sum_{\substack{\hat{y}, \hat{z} \in \hat{Y}_r, \\ d(\hat{y}, \hat{z}) \leq \text{prop}(T_t) + 2r}} \left((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}}) T_t (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}}) \right) \otimes (f(\hat{y}) - f(\hat{z})) \end{aligned} \quad (4.8)$$

Let $v \in H_X \otimes H_Y \otimes \ell^2$ be an arbitrary vector. In the following we will estimate the norm of the vector $[T_t \otimes \text{id}_{\ell^2}, \text{id}_{H_X} \otimes \hat{f}]v$ for large t . For each $\hat{y} \in \hat{Y}_r$, let $v_{\hat{y}} := (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}} \otimes \text{id}_{\ell^2})v$. Then the family $(v_{\hat{y}})_{\hat{y}}$ consists of pairwise orthogonal vectors as $(\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}} \otimes \text{id}_{\ell^2})_{\hat{y}}$ is a family of pairwise orthogonal projections. The sum $\sum_{\hat{y}} v_{\hat{y}}$ converges to v because $\sum_{\hat{y}} \text{id}_{H_X} \otimes 1_{Z_{\hat{y}}} \otimes \text{id}_{\ell^2}$ strongly converges to the identity. Let $t_0 \geq 1$ such that $\text{prop}(T_t) < \delta/2$ for all $t \geq t_0$. In this case, $\text{prop}(T_t) + 2r < \delta/2 + \delta/2 = \delta$. Recall that δ is chosen to ensure $\|f(\hat{y}) - f(\hat{z})\| < \varepsilon$

whenever $d(\hat{y}, \hat{z}) < \delta$. We now obtain the following estimate for all $t \geq t_0$.

$$\begin{aligned}
& \| [T_t \otimes \text{id}_{\ell^2}, \text{id}_{H_X} \otimes \hat{f}] v \|^2 = \\
& \stackrel{(4.8)}{=} \left\| \sum_{\substack{\hat{y}, \hat{z} \in \hat{Y}_r, \\ d(\hat{y}, \hat{z}) \leq \text{prop}(T_t) + 2r}} ((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}}) T_t \otimes (f(\hat{y}) - f(\hat{z}))) v_{\hat{y}} \right\|^2 \\
& = \sum_{\hat{z} \in \hat{Y}_r} \left\| \sum_{\substack{\hat{y} \in \hat{Y}_r, \\ d(\hat{y}, \hat{z}) \leq \text{prop}(T_t) + 2r}} ((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}}) T_t \otimes (f(\hat{y}) - f(\hat{z}))) v_{\hat{y}} \right\|^2 \\
& \leq \sum_{\hat{z} \in \hat{Y}_r} K_{r, \delta} \sum_{\substack{\hat{y} \in \hat{Y}_r, \\ d(\hat{y}, \hat{z}) < \delta}} \| ((\text{id}_{H_X} \otimes 1_{Z_{\hat{z}}}) T_t \otimes (f(\hat{y}) - f(\hat{z}))) v_{\hat{y}} \|^2 \\
& \leq \sum_{\hat{z} \in \hat{Y}_r} K_{r, \delta} \sum_{\substack{\hat{y} \in \hat{Y}_r, \\ d(\hat{y}, \hat{z}) < \delta}} \| T_t \|^2 \underbrace{\| f(\hat{y}) - f(\hat{z}) \|^2}_{< \varepsilon^2} \| v_{\hat{y}} \|^2 \\
& \leq \varepsilon^2 K_{r, \delta} \| T_t \|^2 \underbrace{\sum_{\substack{\hat{z}, \hat{y} \in \hat{Y}_r, \\ d(\hat{z}, \hat{y}) < \delta}} \| v_{\hat{y}} \|^2}_{\leq K_{r, \delta} \| v \|^2} \leq \varepsilon^2 K_{r, \delta}^2 \| T \|^2 \| v \|^2.
\end{aligned}$$

In the first and the last inequality above we use that the number of elements in the set $\hat{Y}_r \cap B_\delta(\hat{z})$ is bounded by $K_{r, \delta}$ by the definition of continuously bounded geometry. We obtain

$$\| [T_t \otimes \text{id}_{\ell^2}, \text{id}_{H_X} \otimes \hat{f}] \| \leq \varepsilon K_{r, \delta} \| T \| = \varepsilon K_{r, 4r} \| T \| \leq \varepsilon (K_4 + 1) \| T \|.$$

Finally, we conclude that for all $t \geq t_0$ the estimate

$$\| [\tau(T_t), \tilde{\rho}_Y(f)] \| \leq \varepsilon (K_4 + 1) \| T \| + 2 \| T \| \underbrace{\| \tilde{\rho}(f) - \hat{f} \|}_{\leq \varepsilon} \leq \varepsilon \| T \| (K_4 + 3)$$

holds. As $\varepsilon > 0$ was arbitrary, this proves the claim. \square

Corollary 4.18. *Suppose that Y has continuously bounded geometry. Let $T \in C_L^*(\rho_{X \times Y})$ and $f \in \bar{\mathfrak{c}}^{\text{red}} Y$. Then*

$$t \mapsto [\tau(T_t), \tilde{\rho}_Y(f)] \in C_0([1, \infty), C^*(\tilde{\rho}_X)). \quad (4.9)$$

Proof. Combine Proposition 4.17 and Lemma 4.5. \square

The first step towards the proof of Theorem 4.13 is to reformulate the E -theoretic slant product in terms of the localization algebra as follows. Consider the map

$$\begin{aligned}
\Upsilon_L: C_L^*(\rho_{X \times Y}) \otimes C_0(Y, \mathcal{K}) &\rightarrow C_L^*(\tilde{\rho}_X) / C_0([1, \infty), C^*(\tilde{\rho}_X)), \\
(T_t)_{t \in [1, \infty)} \otimes f &\mapsto [(\tau(T_t) \circ \tilde{\rho}_Y(f))_{t \in [1, \infty)}].
\end{aligned} \quad (4.10)$$

One verifies readily that the expression $(\tau(T_t) \circ \tilde{\rho}_Y(f))_{t \in [1, \infty)}$ defines an element of the localization algebra $C_L^*(\tilde{\rho}_X)$. Lemma 4.16 implies by the by now familiar argument that Υ_L is a well-defined $*$ -homomorphism.

Lemma 4.19. *Under the isomorphism $\Delta(\rho_{X \times Y})$ and $\Delta(\tilde{\rho}_X)$ from Proposition 4.14, the E -theoretic slant product of Definition 4.11 agrees with the composition*

$$\begin{aligned} \mathbb{K}_p(\mathbb{C}_L^*(X \times Y)) \otimes \mathbb{K}^q(Y) &\cong \mathbb{K}_p(\mathbb{C}_L^*(\rho_{X \times Y})) \otimes \mathbb{K}_{-q}(\mathbb{C}_0(Y, \mathcal{K})) \\ &\xrightarrow{\boxtimes} \mathbb{K}_{p-q}(\mathbb{C}_L^*(\rho_{X \times Y}) \otimes \mathbb{C}_0(Y, \mathcal{K})) \\ &\xrightarrow{(\Upsilon_L)_*} \mathbb{K}_{p-q}(\mathbb{C}_L^*(\tilde{\rho}_X)/\mathbb{C}_0([1, \infty), \mathbb{C}^*(\tilde{\rho}_X))) \\ &\xrightarrow{\cong} \mathbb{K}_{p-q}(\mathbb{C}_L^*(\tilde{\rho}_X)) \cong \mathbb{K}_{p-q}(\mathbb{C}_L^*(X)), \end{aligned}$$

where the third map is the inverse of the isomorphism induced on \mathbb{K} -theory by the canonical projection $\mathbb{C}_L^*(\tilde{\rho}_X) \rightarrow \mathbb{C}_L^*(\tilde{\rho}_X)/\mathbb{C}_0([1, \infty), \mathbb{C}^*(\tilde{\rho}_X))$.

Proof. First of all we notice that the composition in the statement obviously gives the same map as its non-stabilized counterpart, i. e. the analogous composition defined using the $*$ -homomorphism

$$\begin{aligned} \Upsilon'_L : \mathbb{C}_L^*(\rho_{X \times Y}) \otimes \mathbb{C}_0(Y) &\rightarrow \mathbb{C}_L^*(\rho'_X)/\mathbb{C}_0([1, \infty), \mathbb{C}^*(\rho'_X)), \\ (T_t)_{t \in [1, \infty)} \otimes f &\mapsto \left[(T_t \circ (\rho'_Y(f)))_{t \in [1, \infty)} \right] \end{aligned}$$

where $\rho'_X := \rho_X \otimes \text{id}_{H_Y}$ and $\rho'_Y := \text{id}_{H_X} \otimes \rho_Y(f)$ denote the canonical representations of $\mathbb{C}_0(X)$ and $\mathbb{C}_0(Y)$ on $H_{X \times Y}$.

Now the claim is equivalent to commutativity of the diagram

$$\begin{array}{ccc} \mathbb{K}_p(\mathbb{C}_L^*(\rho_{X \times Y})) \otimes \mathbb{K}^q(Y) & \xrightarrow{\boxtimes} & \mathbb{K}_{p-q}(\mathbb{C}_L^*(\rho_{X \times Y}) \otimes \mathbb{C}_0(Y)) \\ \downarrow \Delta(\rho_{X \times Y}) \otimes \text{id} & & \downarrow (\Upsilon'_L)_* \\ E_p(\mathbb{C}_0(X \times Y), \mathbb{C}) \otimes E_{-q}(\mathbb{C}, \mathbb{C}_0(Y)) & & \mathbb{K}_{p-q}(\mathbb{C}_L^*(\rho'_X)/\mathbb{C}_0([1, \infty), \mathbb{C}^*(\rho'_X))) \\ \downarrow / & \swarrow \Delta(\rho'_X) & \uparrow \cong \\ E_{p-q}(\mathbb{C}_0(X), \mathbb{C}) & \xleftarrow{\Delta(\rho'_X)} & \mathbb{K}_{p-q}(\mathbb{C}_L^*(\rho'_X)) \end{array}$$

in which the lower right triangle commutes tautologically.

The vertical left arrows take $x \otimes \theta$ to

$$\delta(\rho_{X \times Y}) \circ (x \boxtimes \text{id}_{\mathbb{C}_0(X \times Y)}) \circ (\text{id}_{\mathbb{C}_0(X)} \boxtimes \theta) = \delta(\rho_{X \times Y}) \circ (x \boxtimes \text{id}_{\mathbb{C}_0(X)} \boxtimes \theta)$$

whereas the composition of the right three arrows of the pentagon map it to

$$\delta(\rho'_X) \circ (((\Upsilon'_L)_* \circ (x \boxtimes \theta)) \boxtimes \text{id}_{\mathbb{C}_0(X)}) = \delta(\rho'_X) \circ ((\Upsilon'_L)_* \boxtimes \text{id}_{\mathbb{C}_0(X)}) \circ (x \boxtimes \theta \boxtimes \text{id}_{\mathbb{C}_0(X)}).$$

Hence it suffices to show that the two asymptotic morphisms $\delta(\rho_{X \times Y})$ and $\delta(\rho'_X) \circ (\Upsilon'_L \otimes \text{id}_{\mathbb{C}_0(X)})$ agree up to precomposing with the homomorphism exchanging the two tensor factors $\mathbb{C}_0(X)$ and $\mathbb{C}_0(Y)$ in the domain. But this holds due to the equation

$$T_t \circ \rho_{X \times Y}(f \otimes g) = T_t \circ \rho'_Y(g) \circ \rho'_X(f). \quad \square$$

Proof of Theorem 4.13. By combining Lemma 4.8 and Corollary 4.18 we get a $*$ -homomorphism

$$\begin{aligned} \tilde{\Upsilon}_L : \mathbb{C}_L^*(\rho_{X \times Y}) \otimes_{\max} \bar{c}^{\text{red}} Y &\rightarrow \mathbb{E}_L^*(\tilde{\rho}_X)/\mathbb{C}_0([1, \infty), \mathbb{C}^*(\tilde{\rho}_X)), \\ (T_t)_{t \in [1, \infty)} \otimes f &\mapsto \left[(\tau(T_t) \circ \tilde{\rho}_Y(f))_{t \in [1, \infty)} \right]. \end{aligned}$$

Its restriction to $C_L^*(\rho_{X \times Y}) \otimes_{\max} C_0(Y, \mathcal{K})$ is Υ_L , see (4.10). Therefore we have the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_L^*(\rho_{X \times Y}) \otimes C_0(Y, \mathcal{K}) & \longrightarrow & C_L^*(\rho_{X \times Y}) \otimes_{\max} \bar{\mathbf{c}}^{\text{red}}(Y) & \longrightarrow & C_L^*(\rho_{X \times Y}) \otimes_{\max} \mathbf{c}^{\text{red}}(Y) \longrightarrow 0 \\
& & \downarrow \Upsilon_L & & \downarrow \tilde{\Upsilon}_L & & \downarrow \Psi_L \\
0 & \longrightarrow & C_L^*(\tilde{\rho}_X)/C_0([1, \infty), C^*(\tilde{\rho}_X)) & \longrightarrow & E_L^*(\tilde{\rho}_X)/C_0([1, \infty), C^*(\tilde{\rho}_X)) & \longrightarrow & E_L^*(\tilde{\rho}_X)/C_L^*(\tilde{\rho}_X) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \parallel \\
0 & \longrightarrow & C_L^*(\tilde{\rho}_X) & \longrightarrow & E_L^*(\tilde{\rho}_X) & \longrightarrow & E_L^*(\tilde{\rho}_X)/C_L^*(\tilde{\rho}_X) \longrightarrow 0,
\end{array}$$

where the arrows directed upwards are the canonical projections. These arrows induce isomorphisms on K-theory because $C_0([1, \infty), C^*(\tilde{\rho}_X))$ is contractible. It induces a diagram in K-theory

$$\begin{array}{ccc}
K_p(C_L^*(\rho_{X \times Y})) \otimes K_{1-q}(\mathbf{c}^{\text{red}}(Y)) & \xrightarrow{\text{id} \otimes \mu^*} & K_p(C_L^*(\rho_{X \times Y})) \otimes K_{-q}(C_0(Y, \mathcal{K})) \\
\downarrow \boxtimes & & \downarrow \boxtimes \\
K_{p+1-q}(C_L^*(\rho_{X \times Y}) \otimes_{\max} \mathbf{c}^{\text{red}}(Y)) & \xrightarrow{\partial} & K_{p-q}(C_L^*(\rho_{X \times Y}) \otimes C_0(Y, \mathcal{K})) \\
\downarrow (\Psi_L)_* & & \downarrow (\Upsilon_L)_* \\
K_{p+1-q}(E_L^*(\tilde{\rho}_X)/C_L^*(\tilde{\rho}_X)) & \xrightarrow{\partial} & K_{p-q}(C_L^*(\tilde{\rho}_X))
\end{array}$$

in which the right hand vertical composition is the E -theoretic slant product up to the isomorphisms Δ by Lemma 4.19 and the composition of the left vertical arrows with the bottom horizontal arrow is $(-1)^p$ times the slant product from Definition 4.9. The lower square of the diagram commutes and the upper square commutes only up to a sign $(-1)^p$ (see Remark 3.2). \square

4.4 Composing slant with external products

In analogy to the first part of Lemma 4.12 we shall now prove the following theorem, which says that the external and slant products which we have constructed are compatible in the sense that $(x \times z)/\theta = x \times (z/\theta)$. It would also be nice to have an analogue of the second part of Lemma 4.12, but this would require the construction of a secondary¹⁶ external product of the form

$$K_{1-q}(\mathbf{c}^{\text{red}}Y) \otimes K_{1-r}(\mathbf{c}^{\text{red}}Z) \rightarrow K_{1-q-r}(\mathbf{c}^{\text{red}}(Y \times Z))$$

and it is completely unclear how such an external product could be constructed.

Theorem 4.20. *The compositions*

$$\begin{array}{ccc}
K_m(X) & \xrightarrow{\times z} & K_{m+p}(X \times Y \times Z) \xrightarrow{/\theta} K_{m+p-q}(X \times Y) \\
S_m(X) & \xrightarrow{\times z} & S_{m+p}(X \times Y \times Z) \xrightarrow{/\theta} S_{m+p-q}(X \times Y) \\
K_m(C^*X) & \xrightarrow{\times z} & K_{m+p}(C^*(X \times Y \times Z)) \xrightarrow{/\theta} K_{m+p-q}(C^*(X \times Y))
\end{array}$$

are equal to the external product with the appropriate slant product z/θ for all $m, p, q \in \mathbb{Z}$ and all z, θ as follows:

¹⁶The external product has to be a secondary one, i. e. one with a degree shift, to make the degrees work out: if x, η, θ have degrees $p, 1-q, 1-r$, respectively, then $(x/\theta)/\eta$ has degree $p-q-r$, and in order to let $x/(\theta \times \eta)$ have the same degree, the external product $\theta \times \eta$ must have degree $1-q-r$ and not $(1-q) + (1-r)$.

- In the first two compositions $z \in K_p(Y \times Z)$ and in the third one $z \in K_p(C^*(Y \times Z))$.
- In the first composition either $\theta \in K^q(Z)$ or $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Z)$ and in the other two $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Z)$.

Proof. For the first of these three compositions and $\theta \in K^q(Z)$ this is exactly the first part of Lemma 4.12. So it remains to show the cases where $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Z)$ for the slant products we have constructed in Section 4.1. We shall only consider the second composition, because everything else goes through completely analogously.

Let ρ_X, ρ_Y, ρ_Z be representations of $C_0(X), C_0(Y), C_0(Z)$ on H_X, H_Y, H_Z , respectively. Denote by $\rho_{Y \times Z}, \rho_{X \times Y \times Z}$ the tensor product representations of $C_0(Y \times Z), C_0(X \times Y \times Z)$ on $H_Y \otimes H_Z, H_X \otimes H_Y \otimes H_Z$, respectively. Furthermore, we define representations $\tilde{\rho}_{X \times Y}$ of $C_0(X \times Y)$ and $\tilde{\rho}_Z$ of $\mathcal{M}(C_0(Z, \mathcal{K}))$ on $H_X \otimes H_Y \otimes H_Z \otimes \ell^2$ in complete analogy to the definition of $\tilde{\rho}_X$ and $\tilde{\rho}_Y$ previously in this section and similarly we define representations $\hat{\rho}_Y$ of $C_0(Y)$ and $\hat{\rho}_Z$ of $\mathcal{M}(C_0(Z, \mathcal{K}))$ on $H_Y \otimes H_Z \otimes \ell^2$.

Note that taking tensor products of operators pointwise in time gives rise to *-homomorphisms

$$\begin{aligned} C_{L,0}^*(\rho_X) \otimes_{\max} E_L^*(\hat{\rho}_Y) &\rightarrow E_{L,0}^*(\tilde{\rho}_{X \times Y}) \\ C_{L,0}^*(\rho_X) \otimes_{\max} C_L^*(\hat{\rho}_Y) &\rightarrow C_{L,0}^*(\tilde{\rho}_{X \times Y}) \end{aligned}$$

with the second one being the restriction of the first one, and hence also a quotient *-homomorphism

$$C_{L,0}^*(\rho_X) \otimes_{\max} \frac{E_L^*(\hat{\rho}_Y)}{C_L^*(\hat{\rho}_Y)} \rightarrow \frac{E_{L,0}^*(\tilde{\rho}_{X \times Y})}{C_{L,0}^*(\tilde{\rho}_{X \times Y})}$$

and these fit into the following diagram.

$$\begin{array}{ccccc} K_m(C_{L,0}^*(\rho_X)) \otimes & \xrightarrow{\boxtimes \text{id}} & K_{m+p} \left(\begin{array}{c} C_{L,0}^*(\rho_X) \\ \otimes_{\max} C_L^*(\rho_{Y \times Z}) \end{array} \right) & \longrightarrow & K_{m+p}(C_{L,0}^*(\rho_{X \times Y \times Z})) \\ \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Z) & & \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Z) & & \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Z) \\ \downarrow \text{id} \otimes \boxtimes & & \downarrow \boxtimes & & \downarrow \boxtimes \\ K_m(C_{L,0}^*(\rho_X)) \otimes & \xrightarrow{\boxtimes} & K_{m+p+1-q} \left(\begin{array}{c} C_{L,0}^*(\rho_X) \\ \otimes_{\max} C_L^*(\rho_{Y \times Z}) \end{array} \right) & \longrightarrow & K_{m+p+1-q} \left(\begin{array}{c} C_{L,0}^*(\rho_{X \times Y \times Z}) \\ \otimes_{\max} \mathfrak{c}^{\text{red}}Z \end{array} \right) \\ K_{p+1-q} \left(\begin{array}{c} C_L^*(\rho_{Y \times Z}) \\ \otimes_{\max} \mathfrak{c}^{\text{red}}Z \end{array} \right) & & & & \\ \downarrow \text{id} \otimes (\Psi_L)_* & & \downarrow (\text{id} \otimes \Psi_L)_* & & \downarrow (\Psi_L)_* \\ K_m(C_{L,0}^*(\rho_X)) \otimes & \xrightarrow{\boxtimes} & K_{m+p+1-q} \left(\begin{array}{c} C_{L,0}^*(\rho_X) \\ \otimes_{\max} \frac{E_L^*(\hat{\rho}_Y)}{C_L^*(\hat{\rho}_Y)} \end{array} \right) & \longrightarrow & K_{m+p+1-q} \left(\begin{array}{c} E_{L,0}^*(\tilde{\rho}_{X \times Y}) \\ \otimes_{\max} \mathfrak{c}^{\text{red}}Z \end{array} \right) \\ K_{p+1-q} \left(\begin{array}{c} E_L^*(\hat{\rho}_Y) \\ \otimes_{\max} C_L^*(\hat{\rho}_Y) \end{array} \right) & & & & \\ \downarrow \text{id} \otimes \partial & & \downarrow \partial & & \downarrow \partial \\ K_m(C_{L,0}^*(\rho_X)) \otimes & \xrightarrow{\boxtimes} & K_{m+p-q} \left(\begin{array}{c} C_{L,0}^*(\rho_X) \\ \otimes C_L^*(\hat{\rho}_Y) \end{array} \right) & \longrightarrow & K_{m+p-q}(C_{L,0}^*(\tilde{\rho}_{X \times Y})) \\ K_{p-q}(C_L^*(\hat{\rho}_Y)) & & & & \end{array}$$

All squares in this diagram commute, except the lower left one, which only commutes up to a sign $(-1)^m$ due to our sign convention in Remark 3.2.

The top and bottom rows are the external products for the localization algebras. The left column is $(-1)^p$ times the slant product while the right column is $(-1)^{m+p}$ times the slant product. The signs match up perfectly, thus proving the claim. \square

Spezializing to the case where Y is a one-point space will give us Property 1.13 (ii) as a corollary, but first we have to introduce the pairings.

Definition 4.21. The pairing

$$\langle -, - \rangle: K_p(C^*Y) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y) \rightarrow K_{p-q}(C^*\{*\}) \cong \begin{cases} \mathbb{Z} & p - q \text{ even} \\ 0 & p - q \text{ odd} \end{cases}$$

is defined as the special case of the slant products for the space X, Y where $X = \{*\}$ is a single point. The same construction applied to the localization algebra instead of the Roe algebra also yields a pairing

$$\langle -, - \rangle: K_p(Y) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y) \rightarrow K_{p-q}(\{*\}) \cong \begin{cases} \mathbb{Z} & p - q \text{ even} \\ 0 & p - q \text{ odd}. \end{cases} \quad \blacklozenge$$

Remark 4.22.

- (i) It is easy to see that the first of these pairings agrees with the one defined in [EM06, Section 6], but possibly only up to a sign $(-1)^p$. This is because Emerson and Meyer have not specified in detail which sign conventions they use (cf. the remarks about signs at the beginning of Section 4.3).
- (ii) Compatibility of the slant products with the assembly maps μ_Y and $\mu_{\{*\}}$ of Y and $\{*\}$ (see Theorem 4.10) shows that the second pairing is a special case of the first one: We have $\langle z, \theta \rangle = \langle \mu_Y(z), \theta \rangle$ for all $z \in K_p(Y)$ and $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Y)$.
- (iii) Recall from the beginning of Section 4.3 that the pairing of K-homology with K-theory is also only a special case of the slant product between them. Therefore, if Y has continuously bounded geometry, then Theorem 4.13 also implies $\langle z, \theta \rangle = \langle z, \mu_Y^*(\theta) \rangle$ for all $z \in K_p(Y)$ and $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Y)$.
- (iv) Note that in the case $X = \{*\}$, the K-theory of $C^*(\rho_X)$ is isomorphic to the K-theory of \mathbb{C} for any non-degenerate faithful representation, because $C^*(\rho_X) \cong \mathcal{K}(H_X)$ with $H_X \neq 0$. Hence we don't have to choose an ample representation for the construction of the pairings, but we may take the non-ample representation of \mathbb{C} on $H_X = \mathbb{C}$ by multiplication instead, simplifying the formulas significantly. \blacklozenge

Corollary 4.23. *The compositions*

$$\begin{array}{ccccc} K_m(X) & \xrightarrow{\times z} & K_{m+p}(X \times Y) & \xrightarrow{/\theta} & K_{m+p-q}(X) \\ S_m(X) & \xrightarrow{\times z} & S_{m+p}(X \times Y) & \xrightarrow{/\theta} & S_{m+p-q}(X) \\ K_m(C^*X) & \xrightarrow{\times z} & K_{m+p}(C^*(X \times Y)) & \xrightarrow{/\theta} & K_{m+p-q}(C^*X) \end{array}$$

are equal to the multiplication with $\langle z, \theta \rangle$, which is either an integer, if $p - q$ is even, or zero by construction, if $p - q$ odd, for all $m, p, q \in \mathbb{Z}$ and all z, θ as follows:

- In the first two compositions $z \in K_p(Y)$ and in the third one $z \in K_p(C^*Y)$.
- In the first composition either $\theta \in K^q(Y)$ or $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Y)$ and in the other two $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Y)$.

Now we also get Property 1.13 (iii) as a corollary.

Corollary 4.24. Denote by $\beta \in K_{1-n}(\mathfrak{c}^{\text{red}}\mathbb{R}^n)$ the Bott element of the Euclidean space, i. e. $\mu^*(\beta) \in K^n(\mathbb{R}^n) \cong \mathbb{Z}$ is the generator which pairs to one with the

fundamental class $[\mathcal{D}_{\mathbb{R}^n}] \in K_n(\mathbb{R}^n)$ of Euclidean space: $\langle [\mathcal{D}_{\mathbb{R}^n}], \beta \rangle = 1$. Then the slant products

$$\begin{aligned} K_{p+n}(X \times \mathbb{R}^n) &\xrightarrow{-/\beta = -/\mu^*(\beta)} K_p(X) \\ S_{p+n}(X \times \mathbb{R}^n) &\xrightarrow{-/\beta} S_p(X) \\ K_{p+n}(C^*(X \times \mathbb{R}^n)) &\xrightarrow{-/\beta} K_p(C^*X) \end{aligned}$$

coincide with the n -fold Mayer–Vietories boundary maps.

Proof. It was shown in [Zei16a, Theorem 5.5] that the external products

$$K_p(X) \xrightarrow{\times[\mathcal{D}_{\mathbb{R}^n}]} K_{p+n}(X \times \mathbb{R}^n) \quad S_p(X), \xrightarrow{\times[\mathcal{D}_{\mathbb{R}^n}]} S_{p+n}(X \times \mathbb{R}^n),$$

which are also called suspension maps, are isomorphisms and that their inverses are given by the n -fold Mayer–Vietoris boundary. Essentially the same proof also shows that the external product

$$K_p(C^*X) \xrightarrow{\times \text{Ind}[\mathcal{D}_{\mathbb{R}^n}]} K_{p+n}(C^*(X \times \mathbb{R}^n))$$

is also an isomorphism whose inverse is given by the n -fold coarse Mayer–Vietoris boundary map. Hence, it suffices to show that the slant products with β are also inverses to these suspension maps.

By the preceding corollary, the composition of the external product followed by the slant product is in each of these three cases equal to multiplication by $\langle [\mathcal{D}_{\mathbb{R}^n}], \beta \rangle = 1$. We conclude that the slant product by β is the left inverse to the suspension map. Because the suspension map is an isomorphism, this suffices to conclude the claim. \square

4.5 Naturality of the slant products

Naturality of our slant products is of course a property that has to be expected. We prove it in this section and use it directly in the next to coarsify all of our previous statements.

Throughout this section let X, X', Y, Y' denote proper metric spaces with Y and Y' having bounded geometry. We consider the Hilbert spaces $H_X, H_Y, H_{X \times Y}, \tilde{H}_X$ and the ample representations $\rho_X, \rho_Y, \rho_{X \times Y}, \tilde{\rho}_X, \tilde{\rho}_Y, \tilde{\rho}_Y$ as before and let the Hilbert spaces $H_{X'}, H_{Y'}, H_{X' \times Y'}, \tilde{H}_{X'}$ and the ample representations $\rho_{X'}, \rho_{Y'}, \rho_{X' \times Y'}, \tilde{\rho}_{X'}, \tilde{\rho}_{Y'}, \tilde{\rho}_{Y'}$ be chosen and constructed in complete analogy.

Definition 4.25 ([HR00b, Definition 6.3.9]). Let $\alpha: X \rightarrow X'$ be a coarse map. An isometry $V: H_X \rightarrow H_{X'}$ covers a coarse map $\alpha: X \rightarrow X'$, if

$$\{(x', \alpha(x)) \in X' \times X' \mid (x', x) \in \text{supp}(V)\}$$

is an entourage of X' . \blacklozenge

Proposition 4.26 ([HR00b, Section 6.3]). Any coarse map $\alpha: X \rightarrow X'$ is covered by an isometry $V: H_X \rightarrow H_{X'}$. Conjugation by V yields a $*$ -homomorphism

$$\text{Ad}_V: C^*(\rho_X) \rightarrow C^*(\rho_{X'}), \quad T \mapsto VTV^*$$

and the induced map in K -theory $(\text{Ad}_V)_*: K_*(C^*X) \rightarrow K_*(C^*X')$ is independent of the choice of covering isometry V and depends functorial on the coarse map α . For this reason we will denote it by α_* . \square

Proposition 4.27 (see [EM06]). *The reduced stable Higson corona—and hence also its K-theory—is contravariantly functorial under coarse maps. If $\beta: Y \rightarrow Y'$ is a coarse map and an element of $[f] \in \mathfrak{c}^{\text{red}}Y'$ is represented by a function $f \in \bar{\mathfrak{c}}^{\text{red}}Y'$, then the class $\beta^*[f] \in \mathfrak{c}^{\text{red}}Y$ is represented by any function $g \in \bar{\mathfrak{c}}^{\text{red}}Y$ for which the (possibly non-continuous) function $g - f \circ \beta$ converges to zero at infinity.* \square

Theorem 4.28. *The slant product for the Roe algebra is natural under pairs of coarse maps $\alpha: X \rightarrow X'$ and $\beta: Y \rightarrow Y'$ in the sense that*

$$\alpha_*(x/\beta^*(\theta)) = (\alpha \times \beta)_*(x)/\theta$$

for all $x \in K_*(C^*(X \times Y))$ and $\theta \in K_*(\mathfrak{c}^{\text{red}}Y')$.

Proof. Assume that $V: H_X \rightarrow H_{X'}$ is an isometry covering α and $W: H_Y \rightarrow H_{Y'}$ is an isometry covering β . Then $V \otimes W: H_{X \times Y} \rightarrow H_{X' \times Y'}$ covers $\alpha \times \beta$ and $V \otimes W \otimes \text{id}_{\ell^2}: \tilde{H}_X \rightarrow \tilde{H}_{X'}$ covers α . The latter is in complete analogy to Lemma 4.4.

We have to show commutativity of the diagram

$$\begin{array}{ccc}
& K_p(C^*(\rho_{X \times Y})) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y') & \\
& \swarrow \text{id} \otimes \beta^* & \searrow (\text{Ad}_{V \otimes W})_* \otimes \text{id} \\
K_p(C^*(\rho_{X \times Y})) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y) & & K_p(C^*(\rho_{X' \times Y'})) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y') \\
& \downarrow & \downarrow \\
& K_{p+1-q}(C^*(\rho_{X \times Y}) \otimes \mathfrak{c}^{\text{red}}Y') & \\
& \swarrow (\text{id} \otimes \beta^*)_* & \searrow (\text{Ad}_{V \otimes W} \otimes \text{id})_* \\
K_{p+1-q}(C^*(\rho_{X \times Y}) \otimes \mathfrak{c}^{\text{red}}Y) & & K_{p+1-q}(C^*(\rho_{X' \times Y'}) \otimes \mathfrak{c}^{\text{red}}Y') \\
& \downarrow \Psi & \downarrow \Psi' \\
K_{p+1-q}(E^*(\tilde{\rho}_X)/C^*(\tilde{\rho}_X)) & \xrightarrow{(\text{Ad}_{V \otimes W \otimes \text{id}})_*} & K_{p+1-q}(E^*(\tilde{\rho}_{X'})/C^*(\tilde{\rho}_{X'})) \\
& \downarrow & \downarrow \\
K_{p-q}(C^*(\tilde{\rho}_X)) & \xrightarrow{(\text{Ad}_{V \otimes W \otimes \text{id}})_*} & K_{p-q}(C^*(\tilde{\rho}_{X'}))
\end{array}$$

where Ψ' is defined in complete analogy to Ψ and we have used that the conjugation by $V \otimes W \otimes \text{id}$ clearly also maps $E^*(\tilde{\rho}_X)$ to $E^*(\tilde{\rho}_{X'})$.

The three quadrilaterals clearly commute. It remains to investigate the pentagon and here it is sufficient to show commutativity of the underlying pentagon of $*$ -homomorphisms:

$$\begin{array}{ccc}
& C^*(\rho_{X \times Y}) \otimes \mathfrak{c}^{\text{red}}Y' & (4.11) \\
& \swarrow \text{id} \otimes \beta^* & \searrow \text{Ad}_{V \otimes W} \otimes \text{id} \\
C^*(\rho_{X \times Y}) \otimes \mathfrak{c}^{\text{red}}Y & & C^*(\rho_{X' \times Y'}) \otimes \mathfrak{c}^{\text{red}}Y' \\
& \downarrow \Psi_L & \downarrow \Psi'_L \\
E^*(\tilde{\rho}_X)/C^*(\tilde{\rho}_X) & \xrightarrow{\text{Ad}_{V \otimes W \otimes \text{id}}} & E^*(\tilde{\rho}_{X'})/C^*(\tilde{\rho}_{X'})
\end{array}$$

Given $S \in C^*(\rho_{X \times Y})$ and $f \in \bar{\mathfrak{c}}^{\text{red}}Y'$, the left path along the pentagon maps $S \otimes [f] \in C^*(\rho_{X \times Y}) \otimes \mathfrak{c}^{\text{red}}Y'$ to the element of $E^*(\tilde{\rho}_{X'})/C^*(\tilde{\rho}_{X'})$ represented by the operator

$$T_1 := (V \otimes W \otimes \text{id}) \circ (S \otimes \text{id}) \circ (V^* \otimes (\bar{\rho}_Y(g) \circ (W^* \otimes \text{id})))$$

with $g \in \bar{\mathfrak{c}}^{\text{red}}Y$ as in Proposition 4.27. The right path, on the other hand, maps it to the element represented by

$$T_2 := (V \otimes W \otimes \text{id}) \circ (S \otimes \text{id}) \circ (V^* \otimes ((W^* \otimes \text{id}) \circ \bar{\rho}_Y(f))).$$

It has to be shown that the difference of these two operators lies in $C^*(\tilde{\rho}_{X'})$.

We already know that choosing different representatives f and g only changes these T_1 and T_2 by elements of $C^*(\tilde{\rho}_{X'})$. Hence it suffices to show that the difference $T_1 - T_2$ can be made arbitrarily small by choosing suitable representatives f and g . This amounts to showing that the difference

$$(W \otimes \text{id}) \circ \bar{\rho}_Y(g) - \bar{\rho}_{Y'}(f) \circ (W \otimes \text{id})$$

can be made arbitrarily small, which will be done in the remaining part of the proof.

Exploiting the bounded geometry of Y we choose $\hat{Y} \subset Y$, $r > 0$ and $K_R > 0$ for each $R > 0$ exactly as in Definition 4.1.(a), that is, such that $Y = \bigcup_{\hat{y} \in \hat{Y}} B_r(\hat{y})$ and for each $R > 0$ the number of elements $\#(\hat{Y} \cap \bar{B}_R(y))$ is bounded by K_R uniformly in y . Furthermore, we choose $\hat{Y}' \subset Y'$ and $r' > 0$ such that the family of balls $\{B_{r'}(\hat{y}')\}_{\hat{y}' \in \hat{Y}'}$ is a locally finite open cover of Y' . This can be done due to the bounded geometry of Y' , although we don't need the full strength of bounded geometry of Y' at this point.

Similar to what we have done in the proof of Lemma 4.5 we choose decompositions of Y and Y' into families $\{Z_{\hat{y}}\}_{\hat{y} \in \hat{Y}}$ and $\{Z'_{\hat{y}'}\}_{\hat{y}' \in \hat{Y}'}$ of pairwise disjoint Borel subsets which are subordinate to the open covers $\{B_r(\hat{y})\}_{\hat{y} \in \hat{Y}}$ and $\{B_{r'}(\hat{y}')\}_{\hat{y}' \in \hat{Y}'}$. Given Borel subsets $Z \subset Y$ or $Z' \subset Y'$, we again denote by $1_Z \in \mathcal{B}(H_Y)$ and $1_{Z'} \in \mathcal{B}(H_{Y'})$ the projections corresponding to the characteristic functions of Z and Z' under the canonical extensions of ρ_Y and $\rho_{Y'}$ to Borel functions, respectively.

Let $s_1 > 0$ be such that $d(\beta(y), \beta(z)) \leq s_1$ for all $y, z \in Y$ with $d(y, z) \leq r$ and define the following number, which is finite, because W covers β :

$$s_2 := \sup\{d(y', \beta(y)) \mid (y', y) \in \text{supp}(W)\}.$$

If $\hat{y} \in \hat{Y}$ and $\hat{y}' \in \hat{Y}'$ are such that $d(\hat{y}', \beta(\hat{y}))$ is bigger than

$$s := s_1 + s_2 + r',$$

then the distance between $Z'_{\hat{y}'}$ and $\beta(Z_{\hat{y}})$ is bigger than s_2 and therefore $1_{Z'_{\hat{y}'}} \circ W \circ 1_{Z_{\hat{y}}} = 0$.

Finally, let $\varepsilon > 0$. We may assume that our element of $\mathfrak{c}^{\text{red}}Y'$ is represented by a function $f \in \bar{\mathfrak{c}}^{\text{red}}Y'$ whose s -variation $\text{Var}_s(f)$ is bounded by ε everywhere. A priori this is only the case outside of a compact subset, but multiplying our function with a slowly varying function $X \rightarrow [0, 1]$ which is 0 on a large compact set and 1 outside of an even larger compact yields a new representative with the demanded property. We can furthermore use a partition of unity $\{\varphi_{\hat{y}}\}_{\hat{y} \in \hat{Y}}$ subordinate to the open cover $\{B_r(\hat{y})\}_{\hat{y} \in \hat{Y}}$ to define the function

$$g := \sum_{\hat{y} \in \hat{Y}} f(\beta(\hat{y}))\varphi_{\hat{y}} \in \bar{\mathfrak{c}}^{\text{red}}Y.$$

Clearly, $g - f \circ \beta$ converges to zero at infinity and g is therefore a valid representative of $\beta^*[f]$.

Now define the strongly convergent series

$$\begin{aligned}\hat{f} &:= \sum_{\hat{y}' \in \hat{Y}'} 1_{Z'_{\hat{y}'}} \otimes f(\hat{y}') \in \mathcal{B}(H_{Y'} \otimes \ell^2), \\ \hat{g} &:= \sum_{\hat{y} \in \hat{Y}} 1_{Z_{\hat{y}}} \otimes f(\beta(\hat{y})) \in \mathcal{B}(H_Y \otimes \ell^2).\end{aligned}$$

It is clear from the the bound on the s -variation of f that $\|\hat{f} - \bar{\rho}_{Y'}(f)\| \leq \varepsilon$ and $\|\hat{g} - \bar{\rho}_Y(g)\| \leq \varepsilon$. Our goal is to estimate the norm of the operator

$$\begin{aligned}\hat{T} &:= (W \otimes \text{id}) \circ \hat{g} - \hat{f} \circ (W \otimes \text{id}) \\ &= \sum_{\substack{\hat{y} \in \hat{Y} \\ \hat{y}' \in \hat{Y}'}} \left(1_{Z'_{\hat{y}'}} \circ W \circ 1_{Z_{\hat{y}}} \right) \otimes (f(\beta(\hat{y})) - f(\hat{y}')) \\ &= \sum_{\substack{\hat{y} \in \hat{Y}, \hat{y}' \in \hat{Y}' \\ d(\hat{y}', \beta(\hat{y})) \leq s}} \left(1_{Z'_{\hat{y}'}} \circ W \circ 1_{Z_{\hat{y}}} \right) \otimes (f(\beta(\hat{y})) - f(\hat{y}')).\end{aligned}$$

Again as in the proof of Lemma 4.5 we decompose an arbitrary $v \in \tilde{H}_X$ into the vectors $v_{\hat{y}} := (\text{id}_{H_X} \otimes 1_{Z_{\hat{y}}} \otimes \text{id}_{\ell^2})v$ with $\hat{y} \in \hat{Y}$ and calculate

$$\begin{aligned}\|\hat{T}v\|^2 &= \sum_{\hat{y}' \in \hat{Y}'} \left\| \sum_{\substack{\hat{y} \in \hat{Y} \\ d(\hat{y}', \beta(\hat{y})) \leq s}} \left(1_{Z'_{\hat{y}'}} \circ W \circ 1_{Z_{\hat{y}}} \right) \otimes (f(\beta(\hat{y})) - f(\hat{y}'))v \right\|^2 \\ &\leq \sum_{\hat{y}' \in \hat{Y}'} (K_s \cdot \varepsilon \cdot \|v_{\hat{y}}\|)^2 = K_s^2 \cdot \varepsilon^2 \cdot \|v\|^2.\end{aligned}$$

Hence we have the norm estimate

$$\|(W \otimes \text{id}) \circ \bar{\rho}_Y(g) - \bar{\rho}_{Y'}(f) \circ (W \otimes \text{id})\| \leq 2\varepsilon + \|\hat{T}\| \leq (2 + K_s)\varepsilon,$$

proving the claim. \square

The proof of Theorem 4.28 can easily be adapted to yield analogous statements for the localization algebras. In order to do so, we first have to recall the functoriality of their K-theories.

Definition 4.29. Suppose that $\alpha: X \rightarrow X'$ is a uniformly continuous coarse map between proper metric spaces and $\rho_X: C_0(X) \rightarrow \mathcal{B}(H_X)$, $\rho_{X'}: C_0(X') \rightarrow \mathcal{B}(H_{X'})$ representations on Hilbert spaces. A uniformly continuous family of isometries $V: [1, \infty) \rightarrow \mathcal{B}(H_X, H_{X'})$, $t \mapsto V_t$ is said to *cover* α if the number

$$\omega_{\alpha, V}(t) := \sup\{d(y, \alpha(x)) \mid (y, x) \in \text{supp}(V_t)\}$$

is finite for all $t \geq 1$ and satisfies $\omega_{\alpha, V}(t) \rightarrow 0$ as $t \rightarrow \infty$. \blacklozenge

Proposition 4.30 (compare [QR10, Proposition 3.2], [WY19, Theorem 6.6.3]). *If $\rho_X: C_0(X) \rightarrow \mathcal{B}(H_X)$ is a representation and $\rho_{X'}: C_0(X') \rightarrow \mathcal{B}(H_{X'})$ an ample one, then any uniformly continuous coarse map $\alpha: X \rightarrow X'$ is covered by a uniformly continuous family of isometries V . Conjugation by V yields a $*$ -homomorphism*

$$\text{Ad}_V: C_L^*(\rho_X) \rightarrow C_L^*(\rho_{X'}), \quad L \mapsto [t \mapsto V_t \circ L_t \circ V_t^*]$$

which gives rise to a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_{L,0}^*(\rho_X) & \longrightarrow & C_L^*(\rho_X) & \longrightarrow & C^*(\rho_X) \longrightarrow 0 \\
& & \downarrow \text{Ad}_V & & \downarrow \text{Ad}_V & & \downarrow \text{Ad}_{V_1} \\
0 & \longrightarrow & C_{L,0}^*(\rho_{X'}) & \longrightarrow & C_L^*(\rho_{X'}) & \longrightarrow & C^*(\rho_{X'}) \longrightarrow 0.
\end{array}$$

The vertical maps in the induced commutative diagram in K-theory

$$\begin{array}{ccccccc}
K_{*+1}(C^*X) & \xrightarrow{\partial} & S_*(X) & \longrightarrow & K_*(X) & \xrightarrow{\text{Ind}} & K_*(C^*X) \\
\downarrow (\text{Ad}_{V_1})_* & & \downarrow (\text{Ad}_V)_* & & \downarrow (\text{Ad}_V)_* & & \downarrow (\text{Ad}_{V_1})_* \\
K_{*+1}(C^*X') & \xrightarrow{\partial} & S_*(X') & \longrightarrow & K_*(X') & \xrightarrow{\text{Ind}} & K_*(C^*X')
\end{array}$$

are independent of the choice of the covering isometry and depend functorial on α . For this reason they will all be denoted by α_* .

Note that this proposition implies the independence of the K-theory of C_L^*X , C_L^*X , C^*X from the chosen ample module which we have mentioned earlier. Furthermore, it includes special cases of the usual functoriality of the K-homology of spaces under proper continuous maps and of the K-theory of the Roe algebra under coarse maps.

Theorem 4.31. *The slant products for the localization algebras are natural under pairs of uniformly continuous coarse maps $\alpha: X \rightarrow X'$ and $\beta: Y \rightarrow Y'$ in the sense that*

$$\alpha_*(x/\beta^*(\theta)) = (\alpha \times \beta)_*(x)/\theta \quad (4.12)$$

for all $x \in K_*(X \times Y)$ or $x \in S_*(X \times Y)$ and $\theta \in K_*(\mathfrak{c}^{\text{red}}Y')$.

Proof. The proof is completely analogous to the one of Theorem 4.28. The main step is to show commutativity of the analogue versions of (4.11) for C_L^* and E_L^* as well as for $C_{L,0}^*$ and $E_{L,0}^*$. To this end, one has to show that the difference of two representatives in $E_L^*(\tilde{\rho}_{X'})$ or $E_{L,0}^*(\tilde{\rho}_{X'})$ differ by an element of $C_L^*(\tilde{\rho}_{X'})$ or $C_{L,0}^*(\tilde{\rho}_{X'})$, respectively. But this is exactly the commutativity of (4.11) applied pointwise to the family of operators at each time in $[1, \infty)$. \square

Remark 4.32. We restricted ourselves in Proposition 4.30 and Theorem 4.31 to functoriality under uniformly continuous coarse maps only, because this is the functoriality which can be described easily in terms of the localization algebras. But both statements of Theorem 4.31 can be generalized further:

- As we have mentioned above, the functoriality of Proposition 4.30 is a special case of the functoriality of the K-homology groups under proper continuous maps under the isomorphisms $K_*(-) \cong K_*(C_L^*(-))$. Hence the results of Section 4.3 together with the naturality of the coassembly map under continuous coarse maps and the well-known naturality of the topological slant product between K-homology and K-theory immediately implies that Formula (4.12) even holds for all $x \in K_*(X \times Y)$, $\theta \in K_*(\mathfrak{c}^{\text{red}}Y')$, proper continuous maps $\alpha: X \rightarrow X'$ and all continuous coarse maps $\beta: Y \rightarrow Y'$, if Y and Y' have continuously bounded geometry.
- By using a different picture of the structure group $S_*(-) = K_*(C_{L,0}^*(-))$ one can show that the functoriality extends to continuous coarse maps [HR00b, Section 12.4], [WY19, Chapter 6]. In this case we can show that Formula (4.12)

even holds for all $x \in S_*(X \times Y)$, $\theta \in K_*(\mathfrak{c}^{\text{red}}Y')$ and continuous coarse maps $\alpha: X \rightarrow X'$ and $\beta: Y \rightarrow Y'$ by applying the following trick:

Given a continuous coarse map $\alpha: X \rightarrow X'$ we define a new proper metric space X'' as the set X equipped with the metric $d'' := \max\{d, \alpha^*d'\}$, that is,

$$d''(x, y) = \max\{d(x, y), d'(\alpha(x), \alpha(y))\},$$

where d and d' are the metrics on X and X' , respectively. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ & \swarrow \alpha' & \nearrow \alpha'' \\ & X'' & \end{array}$$

where α' is the identity map and α'' is the same as α on the underlying sets. It follows from $d'' \geq d$ and $d'' \geq (\alpha'')^*d'$ that both α' and α'' are uniformly continuous coarse maps. Furthermore, the inverse $(\alpha')^{-1}$ is continuous, because α is continuous, and it is a coarse map, because α is a coarse map. Hence, α' is both a homeomorphism and a coarse equivalence and therefore induces an isomorphism between the structure groups. Similarly, the continuous coarse map β can be decomposed into $\beta = \beta'' \circ (\beta')^{-1}$ with β' and β'' being uniformly continuous coarse maps such that $(\beta')^{-1}$ is a continuous coarse map. The naturality of the slant product under the pair (α, β) now follows from the naturality of Theorem 4.31 under the pairs (α', β') and (α'', β'') . \blacklozenge

4.6 Coarsified versions of the external and slant products

In this section we first recall the notions of coarse K-homology KX_* and the coarse K-theory KX^* and define the coarse structure group SX_* . Then we use the results of the previous sections to construct coarsified versions of the external and slant products

$$\begin{aligned} \times: \quad & KX_m(X) \otimes KX_n(Y) && \rightarrow KX_{m+n}(X \times Y) \\ \times: \quad & SX_m(X) \otimes KX_n(Y) && \rightarrow SX_{m+n}(X \times Y) \\ /: \quad & KX_p(X \times Y) \otimes KX^q(Y) && \rightarrow KX_{p-q}(X) \\ /: \quad & KX_p(X \times Y) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y) && \rightarrow KX_{p-q}(X) \\ /: \quad & SX_p(X \times Y) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y) && \rightarrow SX_{p-q}(X) \end{aligned}$$

and show their compatibility with the maps in a coarsified version of the Higson–Roe analytic sequence

$$\cdots \rightarrow K_{*+1}(C^*X) \rightarrow SX_*(X) \rightarrow KX_*(X) \xrightarrow{\mu} K_*(C^*X) \rightarrow \cdots, \quad (4.13)$$

the coarsification maps $K_*(X) \rightarrow KX_*(X)$, $S_*(X) \rightarrow SX_*(X)$ and the co-coarsification map $KX^*(Y) \rightarrow K^*(Y)$, as well as with the co-assembly map $K_{1-q}(\mathfrak{c}^{\text{red}}(Y)) \rightarrow KX^q(Y)$. We also generalize our results obtained in Section 4.4 to the coarsifications.

4.6.1 Definition of the coarse theories

The coarse K-homology KX_* and the coarse structure groups SX_* are defined using the Rips complex construction. We first consider the case of discrete proper metric spaces.

Definition 4.33. Let X be a discrete proper metric space and $R \geq 0$. The Rips complex of X at scale R is (the geometric realization of) the simplicial complex $P_R X$ whose vertex set is X and whose simplices are those spanned by the finite sets of vertices of diameter at most R . \blacklozenge

The Rips complexes can be metrized by proper metrics such that all the inclusions $X \subset P_R X \subset P_S X$ for $0 \leq R < S$ are isometric coarse equivalences. These inclusions turn the Rips complexes into a directed system indexed over $\mathbb{R}_{\geq 0}$ and we can define

$$\mathrm{KX}_*(X) := \varinjlim_{R \geq 0} \mathrm{K}_*(P_R X) \quad \text{and} \quad \mathrm{SX}_*(X) := \varinjlim_{R \geq 0} \mathrm{S}_*(P_R X).$$

If $\alpha: X \rightarrow Y$ is a coarse map between discrete proper metric spaces then for each $R \geq 0$ there exists $S \geq 0$ such that α extends linearly to a continuous coarse map $P_R X \rightarrow P_S Y$ which we denote by the same letter α . If $\beta: X \rightarrow Y$ is another coarse map which is close to α and $\beta: P_R X \rightarrow P_T Y$ is its linear extension to the Rips complex, then the extensions of α and β are homotopic after postcomposing them with the inclusion into $P_U X$ for some $U \gg \max\{S, T\}$. The homotopy is constructed by linear interpolation between α and β and consists of uniformly continuous coarse maps which all belong to the same closeness class. This property together with the homotopy invariance of K-homology and the structure groups¹⁷ immediately implies functoriality of KX_* and SX_* under closeness classes of coarse maps.

Definition 4.34. For any proper metric space X we define

$$\mathrm{KX}_*(X) := \varinjlim_{R \geq 0} \mathrm{K}_*(P_R X') \quad \text{and} \quad \mathrm{SX}_*(X) := \varinjlim_{R \geq 0} \mathrm{S}_*(P_R X')$$

where $X' \subset X$ is any discrete coarsely equivalent subspace. These groups are independent of the choice of discretization $X' \subset X$ up to canonical isomorphism and functorial under closeness classes of coarse maps. \blacklozenge

Note that this coarsening procedure does not yield anything new for the K-theory of the Roe algebra and the stable Higson corona, since the directed systems of Rips complexes consist only of coarse equivalences and therefore $\mathrm{K}_*(C^* X) \cong \varinjlim_{R \geq 0} \mathrm{K}_*(C^*(P_R X'))$ and $\mathrm{K}_*(c^{\mathrm{red}} X) \cong \varprojlim_{R \geq 0} \mathrm{K}_*(c^{\mathrm{red}}(P_R X'))$. The definition of the coarse K-theory KX^* is a bit more complicated (cf. [EM06, Definition 4.3, Note 4.4]) but we note the following.

Lemma 4.35 (cf. [EM06, Remark 4.5]). *For any proper metric space X with discrete coarsely equivalent subspace $X' \subset X$ there is a Milnor- \varprojlim^1 -sequence*

$$0 \rightarrow \varprojlim_{R \geq 0}^1 \mathrm{K}^{*+1}(P_R X') \rightarrow \mathrm{KX}^*(X) \rightarrow \varprojlim_{R \geq 0} \mathrm{K}^*(P_R X') \rightarrow 0$$

which is natural for coarse maps in the obvious way, i. e. the following holds:

If $f: X \rightarrow Y$ is a coarse map and $X' \subset X$, $Y' \subset Y$ are coarsely equivalent discrete subsets, then every coarse map $f': X' \rightarrow Y'$ which is close to the restriction of f to X' induces maps between the left and right terms of the short exact sequences. These induced maps are up to canonical isomorphism independent of the choices of X', Y' and f' , and together with f^ on the middle term they constitute a map between short exact sequences. \square*

¹⁷By Proposition 4.30 we conclude that the structure group is homotopy invariant for uniformly continuous coarse maps. By Remark 4.32 we can extend this to invariance for continuous coarse map, and using (4.13) together with the invariance of $\mathrm{K}_*(C^* X)$ under coarse homotopies [HR94] we can push this even further to invariance under maps which are simultaneously proper continuous and coarse homotopies. But since we do not need this generality here, we will not provide the details of these.

Our slant products will factor through the limit on the right hand side and hence we can ignore the \varprojlim^1 -term for our purposes.

4.6.2 Maps between the coarse theories

Now that we have introduced all the relevant groups, let us describe the maps relating them. We start with the coarsified Higson–Roe exact sequence.

Definition 4.36. The coarsified Higson–Roe sequence (4.13) is obtained as the limit of the Higson–Roe sequences of the Rips complexes $P_R X'$. \blacklozenge

It is again exact and it is clearly natural under coarse maps. Note by the way that the coarse Baum–Connes conjecture for a space X is equivalent to the vanishing of $SX_*(X)$.

Second, there is the coarsified version of the co-assembly map, which we do not define here in order to avoid having to define KX^* precisely. However, we have the following description.

Remark 4.37 ([EM06, Definition 4.6]). For any proper metric space X there is a coarsified co-assembly map

$$\mu^*: K_{1-*}(\mathfrak{c}^{\text{red}} X) \rightarrow KX^*(X)$$

whose composition with the homomorphism $KX^*(X) \rightarrow \varprojlim_{R \geq 0} K^*(P_R X')$ is equal to the limit of the coassembly maps

$$K_{1-*}(\mathfrak{c}^{\text{red}} X) \cong K_{1-*}(\mathfrak{c}^{\text{red}}(P_R X')) \xrightarrow{\mu^*} K^*(P_R X').$$

These maps are all natural under coarse maps. \blacklozenge

Finally, there are also the coarsification and co-coarsification maps, which are defined as follows. Given a proper metric space X , we choose a discrete R -dense subset $X' \subset X$ for some $R > 0$ and a partition of unity $\{\varphi_{x'}\}_{x' \in X'}$ subordinate to the cover $\{B_R(x')\}_{x' \in X'}$ of X . Then the map

$$X \rightarrow P_{2R} X', \quad x \mapsto \sum_{x' \in X'} \varphi_{x'}(x) \cdot x' \tag{4.14}$$

is a continuous coarse equivalence. Furthermore, if a second map $X \rightarrow P_{2S} X''$ is defined in exactly the same way using another S -dense discrete subset X'' and another partition of unity, then the two maps become homotopic via a homotopy of continuous coarse equivalences after postcomposing them with the inclusion into $P_T(X' \cup X'')$ for some $T \gg \max\{2R, 2S\}$. Therefore, the following maps are independent of all choices (using Remark 4.32 to get homotopy invariance of the structure group for continuous coarse maps).¹⁸

Definition 4.38. The coarsification maps

$$\begin{aligned} c: K_*(X) &\rightarrow \varinjlim_{R \geq 0} K_*(P_R X') = KX_*(X) \\ c: S_*(X) &\rightarrow \varinjlim_{R \geq 0} S_*(P_R X') = SX_*(X) \end{aligned}$$

¹⁸Under mild assumptions on the proper metric space X we can even arrange (4.14) to be a uniformly continuous coarse equivalence [BE17, Section 7] and hence we would not need the better homotopy invariance of the structure group discussed in Remark 4.32.

and the co-coarsification map

$$c^*: \mathrm{KX}^*(X) \rightarrow \varprojlim_{R \geq 0} \mathrm{K}^*(P_R X') \rightarrow \mathrm{K}^*(X)$$

are defined as the maps induced by any map $X \rightarrow P_{2R} X'$ as above. \blacklozenge

Lemma 4.39. *The coarsification and co-coarsification maps are natural under continuous coarse maps.*

Proof. Assume that a diagram

$$\begin{array}{ccc} X & \supseteq & X' \\ \alpha \downarrow & & \downarrow \alpha' \\ Y & \supseteq & Y' \end{array}$$

is given, where α is a continuous coarse map between proper metric spaces and α' is a coarse map between discrete coarsely equivalent subsets such that the diagram commutes up to closeness. Then for each sufficiently large $R \geq 0$ there is $S \geq 0$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & P_{2R} X' \\ \alpha \downarrow & & \downarrow \alpha' \\ Y & \longrightarrow & P_{2S} Y' \end{array}$$

of continuous coarse maps commutes up to closeness. If S is chosen large enough, then the diagram even commutes up to a homotopy via continuous coarse maps which are close to α , the homotopy being defined by a linear interpolation. The claim follows. \square

The following important property follows immediately from the definition.

Lemma 4.40. *The coarse assembly and co-assembly maps decompose into the compositions of their coarsified counterparts and the (co-)coarsification maps, i. e. the diagrams*

$$\begin{array}{ccc} \mathrm{K}_*(X) & \xrightarrow{\mu} & \mathrm{K}_*(C^* X) \\ & \searrow c & \nearrow \mu \\ & \mathrm{KX}_*(X) & \end{array} \qquad \begin{array}{ccc} \mathrm{K}_{1-*}(c^{\mathrm{red}} X) & \xrightarrow{\mu^*} & \mathrm{K}^*(X) \\ & \searrow \mu^* & \nearrow c^* \\ & \mathrm{KX}^*(X) & \end{array}$$

commute.

Let us recall for future reference the well-known conditions under which the (co-)coarsification maps are isomorphisms. To this end we first recall the following definition:

Definition 4.41. Let X be a metric space. We call X *uniformly contractible*, if for every $R > 0$ there exists an $S \geq R$ such that for every point $x \in X$ the inclusion $B_R(x) \hookrightarrow B_S(x)$ is nullhomotopic. \blacklozenge

The proof of the following result can be found in several places in the literature like [Roe96, Chapter 2], [BE16, Proposition 6.105], [NY12, Theorem 7.6.2] or [EM06, Theorem 4.8].

Proposition 4.42. *Let X be a proper metric space of bounded geometry. If X is uniformly contractible, then the (co-)coarsification maps are isomorphisms.*

4.6.3 The coarsified external and slant products

The construction of the coarsified external and slant products relies on the following easy lemma, which essentially says that the products of Rips complexes $P_R X' \times P_R Y'$ can be seen as deformation retracts of the Rips complex $P_R(X' \times Y')$ of the products, up to enlarging the scale of the Rips complexes.

Lemma 4.43. *Let X' and Y' be discrete proper metric spaces. For every $R \geq 0$ we define the continuous coarse equivalences*

$$\begin{aligned} p_R &: P_R(X' \times Y') \rightarrow P_R X' \times P_R Y' \\ i_R &: P_R X' \times P_R Y' \rightarrow P_{2R}(X' \times Y') \end{aligned}$$

by the formulas

$$\begin{aligned} p_R \left(\sum_{(x',y') \in X' \times Y'} \nu_{(x',y')} \cdot (x', y') \right) &= \left(\sum_{x' \in X'} \sum_{y' \in Y'} \nu_{(x',y')} \cdot x', \sum_{y' \in Y'} \sum_{x' \in X'} \nu_{(x',y')} \cdot y' \right) \\ i_R \left(\sum_{x' \in X'} \kappa_{x'} \cdot x', \sum_{y' \in Y'} \lambda_{y'} \cdot y' \right) &= \sum_{(x',y') \in X' \times Y'} \kappa_{x'} \lambda_{y'} \cdot (x', y'). \end{aligned}$$

Then $p_{2R} \circ i_R$ is equal to the inclusion $P_R X' \times P_R Y' \rightarrow P_{2R} X' \times P_{2R} Y'$ and the composition $i_R \circ p_R$ is homotopic to the inclusion $P_R(X' \times Y') \rightarrow P_{2R}(X' \times Y')$ via a homotopy of continuous coarse maps which are all close to the inclusion map. \square

Corollary 4.44. *Given proper metric spaces X and Y with discrete coarsely equivalent subsets $X' \subset X$ and $Y' \subset Y$, the maps p_R and i_R give rise to natural isomorphisms*

$$\begin{aligned} \mathrm{KX}_*(X \times Y) &= \varinjlim_{R \geq 0} \mathrm{K}_*(P_R(X' \times Y')) \cong \varinjlim_{R \geq 0} \mathrm{K}_*(P_R X' \times P_R Y'), \\ \mathrm{SX}_*(X \times Y) &= \varinjlim_{R \geq 0} \mathrm{S}_*(P_R(X' \times Y')) \cong \varinjlim_{R \geq 0} \mathrm{S}_*(P_R X' \times P_R Y'). \end{aligned}$$

The existence of the coarsified external and slant products are now a direct consequence of this corollary and the naturality of the external and slant products under the pairs of inclusions $P_R X' \subset P_S X'$ and $P_R Y' \subset P_S Y'$ for $R \leq S$.

Definition 4.45. The coarsified external products

$$\begin{aligned} \times &: \mathrm{KX}_m(X) \otimes \mathrm{KX}_n(Y) \rightarrow \mathrm{KX}_{m+n}(X \times Y) \\ \times &: \mathrm{SX}_m(X) \otimes \mathrm{KX}_n(Y) \rightarrow \mathrm{SX}_{m+n}(X \times Y) \end{aligned}$$

are obtained by taking the direct limit over the external products

$$\begin{aligned} \times &: \mathrm{K}_m(P_R X') \otimes \mathrm{K}_n(P_R Y') \rightarrow \mathrm{K}_{m+n}(P_R X' \times P_R Y') \\ \times &: \mathrm{S}_m(P_R X') \otimes \mathrm{K}_n(P_R Y') \rightarrow \mathrm{S}_{m+n}(P_R X' \times P_R Y') \end{aligned}$$

and if Y has bounded geometry then the coarsified slant products

$$\begin{aligned} / &: \mathrm{KX}_p(X \times Y) \otimes \mathrm{KX}^q(Y) \rightarrow \mathrm{KX}_{p-q}(X) \\ / &: \mathrm{KX}_p(X \times Y) \otimes \mathrm{K}_{1-q}(\mathbf{c}^{\mathrm{red}} Y) \rightarrow \mathrm{KX}_{p-q}(X) \\ / &: \mathrm{SX}_p(X \times Y) \otimes \mathrm{K}_{1-q}(\mathbf{c}^{\mathrm{red}} Y) \rightarrow \mathrm{SX}_{p-q}(X) \end{aligned}$$

are obtained by taking the direct limit over the slant products

$$\begin{aligned}
/ : K_p(P_R X' \times P_R Y') \otimes KX^q(Y) &\rightarrow K_p(P_R X' \times P_R Y') \otimes K^q(P_R Y') \\
&\rightarrow K_{p-q}(P_R X') \\
/ : K_p(P_R X' \times P_R Y') \otimes K_{1-q}(\mathfrak{c}^{\text{red}} Y) &\cong K_p(P_R X' \times P_R Y') \otimes K_{1-q}(\mathfrak{c}^{\text{red}}(P_R Y')) \\
&\rightarrow K_{p-q}(P_R X') \\
/ : S_p(P_R X' \times P_R Y') \otimes K_{1-q}(\mathfrak{c}^{\text{red}} Y) &\cong S_p(P_R X' \times P_R Y') \otimes K_{1-q}(\mathfrak{c}^{\text{red}}(P_R Y')) \\
&\rightarrow S_{p-q}(P_R X').
\end{aligned}$$

In the special case that X is a single point we obtain the coarsified versions of the pairings

$$\begin{aligned}
\langle -, - \rangle : KX_p(Y) \otimes KX^q(Y) &\rightarrow K_{p-q}(\mathbb{C}) \\
\langle -, - \rangle : KX_p(Y) \otimes K_{1-q}(\mathfrak{c}^{\text{red}} Y) &\rightarrow K_{p-q}(\mathbb{C})
\end{aligned}$$

whose values lie in \mathbb{Z} if $p - q$ is even and who vanish if $p - q$ is odd. \blacklozenge

The following properties are obvious by applying the properties of the uncoarsified external and slant products proven in Sections 4.2, 4.3, 4.4, 4.5 to the Rips complexes and taking limits.

Theorem 4.46. *The coarsified external and slant products are natural for pairs of coarse maps and compatible with the maps in the coarsified Higson–Roe sequence (4.13), the coarsification maps $K_*(X) \rightarrow KX_*(X)$, $S_*(X) \rightarrow SX_*(X)$ and co-coarsification map $KX^*(Y) \rightarrow K^*(Y)$, as well as the co-assembly map $K_{1-q}(\mathfrak{c}^{\text{red}} Y) \rightarrow KX^q(Y)$.¹⁹*

Further, taking first the external product with an element $z \in KX_m(Y \times Z)$ and then the slant product with an element $\theta \in K_{1-n}(\mathfrak{c}^{\text{red}} Z)$ or $\theta \in KX^n(Z)$ is equal to the external product with $z/\theta \in KX_{m-n}(Y)$, and in particular if $Y = \{*\}$ is a one-point space, then this composition is equal to multiplication with $\langle z, \theta \rangle = \langle z, \mu^*(\theta) \rangle = \langle \mu(z), \theta \rangle$. \square

5 Equivariant slant products

We now generalize the results from the previous section to an equivariant setup. Throughout this section let X, Y be proper metric spaces and let G, H be countable discrete groups acting properly and isometrically on X and Y , respectively. Furthermore we assume that Y has bounded geometry. The cases where Y is required to even have continuously bounded geometry will be pointed out explicitly.

We have already seen in Section 3.2 what the equivariant Higson–Roe sequence

$$\dots \rightarrow K_{*+1}(C_G^* X) \rightarrow S_*^G(X) \rightarrow K_*^G(X) \xrightarrow{\text{Ind}} K_*(C_G^* X) \rightarrow \dots$$

is and how the equivariant external products with elements of $K_*^H(Y)$ and $K_*(C_H^* Y)$ are constructed. This theory is already well established.

On the cohomological side we use crossed products to define equivariant analogues of $K^*(Y)$ and $K_*(\mathfrak{c}^{\text{red}} Y)$. First of all, we define the equivariant K -theory of Y as

$$K_H^*(Y) := K_{-*}(C_0(Y) \rtimes H).$$

¹⁹For compatibility with the co-assembly map we have to note that if Y has bounded geometry, then the Rips complex of any uniformly discrete, coarsely equivalent subset of it has continuously bounded geometry since it will be a simplicial complex of bounded geometry; see Definition 4.1.

This definition is justified by [BHS10, Theorem 6.8] which says that $K_H^0(Y)$ is naturally isomorphic to the Grothendieck group of H -equivariant vector bundles on Y if the action of H on Y is cocompact. Moreover, if the action of H on Y is free, then it is well known that $C_0(Y) \rtimes H$ is Morita equivalent to $C_0(H \backslash Y)$ (see also Section 5.5 below) and therefore $K_H^*(Y) \cong K^*(H \backslash Y)$.

Note that since we assume H to act properly on Y , it follows from [Ech17, Rem. 4.16] that the maximal crossed product of $C_0(Y)$ by H coincides with the reduced one, and consequently also with any other. Hence we only write $C_0(Y) \rtimes H$.

At first sight it might look tempting to define $K_H^*(Y)$ differently²⁰ as the E -theory group $E_{-*}^H(\mathbb{C}, C_0(Y))$ in order to define a slant product

$$K_p^{G \times H}(X \times Y) \otimes K_H^q(Y) \rightarrow K_{p-q}^G(X) \quad (5.1)$$

as in Definition 4.11 via E -theoretic products, but there is one big problem with this attempt: There simply is no E -theoretic product which gets rid of the H -equivariance as is needed for (5.1). The problem seems to be that both entries of equivariant E -theory have to be C^* -algebras which are being acted on by the same group, although we would prefer to consider \mathbb{C} without any action in this case. Perhaps it is possible to fix this issue by generalizing the notion of equivariant E -theory groups to allow for different equivariances in the two entries.

Question 5.1. *Let A be a G - C^* -algebra, B be a H - C^* -algebra and α be a homomorphism between G and H . Is there a meaningful notion of E -theory groups $E_*^\alpha(A, B)$ which specialize to $E_*^G(A, B)$ if $G = H$ and α is the identity?*

Instead of going the E -theory path, we circumvent this problem by defining the equivariant slant product (5.1) in Definition 5.22 below for Y of continuously bounded geometry ad hoc by turning an equivariant version of Lemma 4.19 into a definition.

In order to construct the equivariant analogues of the other slant products, we use $K_*(\mathfrak{c}^{\text{red}}Y \rtimes_\mu H)$ as the equivariant analogue of $K_*(\mathfrak{c}^{\text{red}}Y)$, where \rtimes_μ denotes any exact crossed product functor in the sense of [BGW16, Definition 3.1].

A natural choice for an exact crossed product functor is to use the minimal exact and Morita compatible crossed product functor $\rtimes_{\mathcal{E}}$ used in the recent reformulation of the Baum–Connes conjecture due to Baum, Guentner and Willett [BGW16, Definition 4.1]. Note that it was proven by Buss, Echterhoff and Willett [BEW18b] that the minimal exact and Morita compatible crossed product functor is the same as the minimal exact one.

Since $\mathfrak{c}^{\text{red}}Y$ is in general not separable, another natural choice for an exact crossed product functor is the minimal exact correspondence crossed product functor²¹, which only for separable H - C^* -algebras is known to coincide with the minimal exact crossed product [BEW18a, Corollary 8.13]. Actually, in Section 5.1 we will see that the minimal exact correspondence crossed product functor is indeed a better choice.

Before we construct these slant products in Section 5.3 and prove their properties in Section 5.4, let us first explain in the next two sections how the group $K_*(\mathfrak{c}^{\text{red}}Y \rtimes_\mu H)$ appears in the theory of equivariant co-assembly and show that it contains sufficiently many elements for our purposes.

²⁰Not only the definitions are different, but there is apparently not even an isomorphism between them.

²¹There are several equivalent characterizations of correspondence crossed product functors [BEW17, Thm 4.5]. The main property of them that we will need is that they admit a descent homomorphism in Kasparov theory [BEW17, Thm 4.9].

5.1 Equivariant co-assembly maps

We have already mentioned that the maximal crossed product $C_0(Y) \rtimes_{\max} H$ coincides with the reduced crossed product $C_0(Y) \rtimes_{\text{red}} H$ and hence also with any other crossed product. As both the maximal and the reduced crossed product are Morita compatible, the crossed products $C_0(Y, \mathcal{K}) \rtimes_{\max} H$ and $C_0(Y, \mathcal{K}) \rtimes_{\text{red}} H$ are both isomorphic to $(C_0(Y) \rtimes H) \otimes \mathcal{K}$ and hence the same is true for any other crossed product $C_0(Y, \mathcal{K}) \rtimes_{\mu} H$. It follows that $K_*(C_0(Y, \mathcal{K}) \rtimes_{\mu} H) \cong K_H^{-*}(Y)$ for all crossed product functors \rtimes_{μ} .

Now, for any exact crossed product \rtimes_{μ} we have a short exact sequence

$$0 \rightarrow C_0(Y, \mathcal{K}) \rtimes_{\mu} H \rightarrow \bar{\mathfrak{c}}^{\text{red}}Y \rtimes_{\mu} H \rightarrow \mathfrak{c}^{\text{red}}Y \rtimes_{\mu} H \rightarrow 0,$$

whose connecting homomorphism

$$\mu_H^* : K_*(\mathfrak{c}^{\text{red}}Y \rtimes_{\mu} H) \rightarrow K_H^{1-*}(Y)$$

is an equivariant version of the coarse co-assembly map.

Emerson and Meyer constructed in [EM07], see also [EM08, Section 2.3], the co-assembly map

$$\mu_{\text{EM}}^* : K_*^{\text{top}}(H, \mathfrak{c}^{\text{red}}Y) \rightarrow K_H^{1-*}(Y),$$

where the left hand side is defined as in the Baum–Connes conjecture for the coefficient C^* -algebra $\mathfrak{c}^{\text{red}}Y$. The Baum–Connes assembly map is, with these coefficients and using \rtimes_{μ} , a map $\mu_*^{\text{BC}} : K_*^{\text{top}}(H, \mathfrak{c}^{\text{red}}Y) \rightarrow K_*(\mathfrak{c}^{\text{red}}Y \rtimes_{\mu} H)$.²² We can therefore form the following diagram:

$$\begin{array}{ccc} K_*^{\text{top}}(H, \mathfrak{c}^{\text{red}}Y) & \xrightarrow{\mu_{\text{EM}}^*} & K_H^{1-*}(Y) \\ & \searrow \mu_*^{\text{BC}} & \nearrow \mu_H^* \\ & K_*(\mathfrak{c}^{\text{red}}Y \rtimes_{\mu} H) & \end{array} \quad (5.2)$$

Lemma 5.2. *If \rtimes_{μ} is an exact correspondence crossed product functor,²³ then the Diagram (5.2) commutes.*

Proof. This lemma is true more or less directly by definition of the map μ_{EM}^* as given in [EM07]. Because they are working exclusively with the maximal, resp. reduced crossed product, the only thing we have to do is to make sure that all their constructions and arguments work equally well with any exact correspondence crossed product functor.

By [EM07, Definition 12] we have the following commutative diagram

$$\begin{array}{ccc} K_*((\mathfrak{c}^{\text{red}}Y \otimes_{\max} P) \rtimes_{\mu} H) & \xrightarrow{\partial} & K_{*-1}((C_0(Y) \otimes_{\max} P) \rtimes_{\mu} H) \\ \cong \updownarrow & & \cong \downarrow D_* \\ K_*^{\text{top}}(H, \mathfrak{c}^{\text{red}}Y) & \xrightarrow{\mu_{\text{EM}}^*} & K_H^{1-*}(Y) \end{array}$$

which is used to define the map μ_{EM}^* . Here ∂ is the boundary map induced by a certain short exact sequence of C^* -algebras (this is why we need the crossed product

²²Note the slight technicality here that $\mathfrak{c}^{\text{red}}Y$ is in general not separable. We are working here with the convention that in the non-separable case we just take everywhere directed limits over the separable sub- C^* -algebras.

²³Note that it must be defined for non-separable C^* -algebras. One way to get such a functor is to use again directed limits over separable C^* -algebras [BEW18a, Lem. 8.11].

functor \rtimes_{μ} to be exact), and \mathbf{P} is a certain H - C^* -algebra which supports the so-called Dirac morphism $\mathbf{D} \in \mathrm{KK}^H(\mathbf{P}, \mathbb{C})$. It is known that the Baum–Connes assembly map with coefficients in a C^* -algebra A and using \rtimes_{μ} is equivalent to the map $\mathbf{D}_*: K_*((A \otimes_{\max} \mathbf{P}) \rtimes_{\mu} H) \rightarrow K_*(A \rtimes_{\mu} H)$, and the left vertical isomorphism in the previous diagram is the one which identifies the corresponding domains [MN06, Theorem 5.2]. This isomorphism is independent of the crossed product used [EM07, Theorem 22]. Note that here we need that the crossed product is compatible with the descent homomorphism (for instance if it is a correspondence crossed product functor). That the right vertical map in the above diagram is an isomorphism is explained in [EM07, Beginning of Section 2.7], and we are using that the domain of \mathbf{D}_* is independent of the crossed product used [EM07, Theorem 22].

Applying \mathbf{D}_* to the boundary map in the above diagram, we get

$$\begin{array}{ccc} K_*(\mathfrak{c}^{\mathrm{red}}Y \rtimes_{\mu} H) & \xrightarrow{\partial} & K_H^{1-*}(Y) \\ \uparrow \mathbf{D}_* & & \cong \uparrow \mathbf{D}_* \\ K_*((\mathfrak{c}^{\mathrm{red}}Y \otimes_{\max} \mathbf{P}) \rtimes_{\mu} H) & \xrightarrow{\partial} & K_*((C_0(Y) \otimes_{\max} \mathbf{P}) \rtimes_{\mu} H) \end{array}$$

which commutes because Kasparov products are compatible with boundary maps induced from short exact sequences.

The top horizontal map in the last diagram coincides with μ_H^* , and the composition

$$K_*^{\mathrm{top}}(H, \mathfrak{c}^{\mathrm{red}}Y) \xleftarrow{\cong} K_*((\mathfrak{c}^{\mathrm{red}}Y \otimes_{\max} \mathbf{P}) \rtimes_{\mu} H) \xrightarrow{\mathbf{D}_*} K_*(\mathfrak{c}^{\mathrm{red}}Y \rtimes_{\mu} H)$$

identifies with the Baum–Connes assembly map μ_*^{BC} , which is a property of the Dirac morphism \mathbf{D}_* ; see [MN06, Section 4–6] or [EM07, Theorem 22]. \square

Commutativity of Diagram (5.2) implies the next corollary stating that in many cases we have enough elements to take equivariant slant products.

Before we go into the corollary, let us recall that $\underline{E}H$ denotes the classifying space for proper H -actions. We say that it is H -finite, if it consists only of finitely many H -orbits of cells. Models for $\underline{E}H$ are unique up to equivariant homotopy, which implies that if we have two H -finite models $\underline{E}H$ and $\underline{E}H'$, then

$$K_H^*(\underline{E}H) \cong K_H^*(\underline{E}H').$$

Furthermore, if we put any equivariant length metric on the H -finite model $\underline{E}H$ and pick any point $p \in \underline{E}H$, then the map $H \rightarrow \underline{E}H, h \mapsto hp$ is a coarse equivalence which is H -equivariant. Note that in general a coarse inverse $\underline{E}H \rightarrow H$ cannot be equivariant, because the action of H on $\underline{E}H$ is not always free, but the above properties nevertheless ensure that the induced map $\mathfrak{c}^{\mathrm{red}}\underline{E}H \rightarrow \mathfrak{c}^{\mathrm{red}}H$ is an isomorphism of C^* -algebras which is also H -equivariant, i.e. an isomorphism of H - C^* -algebras. This isomorphism is even canonical, because the coarse equivalences $H \rightarrow \underline{E}H$ are pairwise close for different choices of the point $p \in \underline{E}H$. Consequently, we obtain a canonical isomorphism

$$K_*(\mathfrak{c}^{\mathrm{red}}H \rtimes_{\mu} H) \cong K_*(\mathfrak{c}^{\mathrm{red}}\underline{E}H \rtimes_{\mu} H) \quad (5.3)$$

for each H -finite model for $\underline{E}H$.

In addition, any H -equivariant homotopy equivalence between two H -finite models $\underline{E}H$ and $\underline{E}H'$ is automatically a quasi-isometry. Hence it induces a canonical isomorphism $K_*(\mathfrak{c}^{\mathrm{red}}\underline{E}H \rtimes_{\mu} H) \cong K_*(\mathfrak{c}^{\mathrm{red}}\underline{E}H' \rtimes_{\mu} H)$ which is compatible with (5.3).

Everything that has been said in the above paragraphs tells us that the co-assembly map in the following corollaries is canonical.

Corollary 5.3. *Let H be a countable discrete group that admits a H -finite classifying space for proper H -actions $\underline{E}H$.*

If H has a γ -element, then the equivariant coarse co-assembly map

$$\mu_H^*: K_*(\mathfrak{c}^{\text{red}}H \rtimes_{\mu} H) \rightarrow K_H^{1-*}(\underline{E}H)$$

is surjective for any exact correspondence crossed product functor \rtimes_{μ} .

Proof. We consider Diagram (5.2) for the group H and the space $\underline{E}H$. It follows from [EM08, Proposition 13] that μ_{EM}^* is an isomorphism since H has a γ -element. Hence by Diagram (5.2) the map μ_H^* is surjective. \square

Let \rtimes_{μ} be a correspondence crossed product functor and assume that H admits a γ -element. Then the assembly map $\mu_*^{\text{BC}}: K_*^{\text{top}}(H, A) \rightarrow K_*(A \rtimes_{\mu} H)$ is an isomorphism onto $\gamma \cdot K_*(A \rtimes_{\mu} H)$ for every H - C^* -algebra A , where γ acts as a projection via the Kasparov product.²⁴ Hence Corollary 5.3 refines to the statement that

$$\mu_H^*: \gamma \cdot K_*(\mathfrak{c}^{\text{red}}H \rtimes_{\mu} H) \rightarrow K_H^{1-*}(\underline{E}H)$$

is an isomorphism. Hence, if γ acts as the identity, then μ_H^* is an isomorphism.

Corollary 5.4. *Let H be a countable discrete group that admits a H -finite classifying space for proper H -actions $\underline{E}H$. Assume further that H is exact and that it satisfies (the reduced version of) the Baum–Connes conjecture with coefficients.*

Then the equivariant coarse co-assembly map

$$\mu_H^*: K_*(\mathfrak{c}^{\text{red}}H \rtimes_{\text{red}} H) \rightarrow K_H^{1-*}(\underline{E}H)$$

is an isomorphism, where \rtimes_{red} is the reduced crossed product functor.

Proof. If H is exact, then it has a γ -element and the reduced crossed product is exact (and it is always a correspondence crossed product). Therefore the claim follows from the discussion prior to the corollary, since satisfying (the reduced version of) the Baum–Connes conjecture with coefficients gives that γ acts as the identity on $K_*(\mathfrak{c}^{\text{red}}H \rtimes_{\text{red}} H)$. \square

Example 5.5. Gromov hyperbolic groups satisfy the assumptions of Corollary 5.4. For them $\underline{E}H$ can be taken as the Rips complex $P_R(H)$ of H for a large enough $R \gg 1$ ([BCH94, Section 2], [MS02]), the Baum–Connes conjecture with coefficients was proven by Lafforgue [Laf12; Pus12], and exactness follows from them having finite asymptotic dimension ([Gro93, Page 23][Roe05]). \blacklozenge

5.2 Exactness of groups and the stable Higson corona

In the previous sections we saw that the choice of crossed product $\mathfrak{c}^{\text{red}}Y \rtimes_{\mu} H$ matters, and that we have a connection to exactness of the group H . In the present section we will investigate this connection further, and relate it to the so-called *weak containment* property of $\mathfrak{c}^{\text{red}}Y$, i.e., the question in which cases we have $\mathfrak{c}^{\text{red}}Y \rtimes_{\text{max}} H \cong \mathfrak{c}^{\text{red}}Y \rtimes_{\text{red}} H$. Most of the results here were developed in discussions with Rufus Willett.²⁵

²⁴In order to define this action we need that \rtimes_{μ} admits a suitable descent homomorphism, which is the case for correspondence crossed product functors [BEW18a, Proposition 6.1].

²⁵One can prove similar results for $\mathfrak{c}Y$ instead of $\mathfrak{c}^{\text{red}}Y$. But since we are mainly using $\mathfrak{c}^{\text{red}}Y$ in this paper, we have restricted our attention to it in this section.

5.2.1 The case of the (stable) Higson compactification

Before we can prove the main result of this section (Proposition 5.8 below), we first need a technical result about the double dual of the stable Higson compactification $\bar{c}^{\text{red}}Y$ of a metric space Y and its relation to the (usual) Higson compactification of Y . Recall that the *Higson compactification* is the compact space corresponding to $C_h(Y)$, the unital commutative C^* -algebra of all bounded, continuous functions $Y \rightarrow \mathbb{C}$ of vanishing variation. The *Higson corona* $\partial_h Y$ is the compact space defined via $C(\partial_h Y) := C_h(Y)/C_0(Y)$.

Lemma 5.6. *Let Y be a proper metric space.*

Then there is an embedding of $C_h(Y)$ into the center of the double dual of $\bar{c}^{\text{red}}Y$, i. e., an injective and unital $$ -homomorphism*

$$C_h(Y) \rightarrow Z((\bar{c}^{\text{red}}Y)^{**}). \quad (5.4)$$

If Y is further equipped with an isometric action²⁶ of a countable discrete group H , then the map (5.4) is equivariant for the induced actions of H on the corresponding C^ -algebras.*

Proof. Recall that the Banach space dual of the compact operators $\mathcal{K}(H)$ are the trace class operators $\mathcal{S}^1(H)$, and that the Banach space dual of $\mathcal{S}^1(H)$ is $\mathcal{B}(H)$, and hence $\mathcal{K}(H)^{**} \cong \mathcal{B}(H)$ [Bla06, Theorem I.8.6.1]. Further, $\mathcal{B}(H)$ can be equipped with the ultra-weak topology²⁷ whose restriction to the closed unit ball of $\mathcal{B}(H)$ coincides with the corresponding restriction of the weak operator topology [Mur90, Theorem 4.2.4]. Hence, choosing an orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of H and setting $p_n \in \mathcal{K}(H)$ to be the orthogonal projection onto the linear span of e_1, \dots, e_n , we get a sequence $(p_n)_{n \in \mathbb{N}}$ of compact operators of norm 1 converging in the weak operator topology to the identity operator id_H on H . Consequently, $(p_n)_{n \in \mathbb{N}}$ converges ultra-weakly to id_H .

Now choose any sequence of compact operators $(k_n)_{n \in \mathbb{N}}$ on ℓ^2 converging ultra-weakly to the identity id_{ℓ^2} . The sequence $(f \otimes k_n)_{n \in \mathbb{N}}$ for $f \in C_h(Y)$, which is a sequence in $\bar{c}^{\text{red}}Y \subset (\bar{c}^{\text{red}}Y)^{**}$, converges ultra-weakly in $(\bar{c}^{\text{red}}Y)^{**}$. Its limit is, by definition, the image of f under the sought map (5.4).

It is clear that (5.4) is an injective and unital $*$ -homomorphism. It remains to show that its image is contained in the center of $(\bar{c}^{\text{red}}Y)^{**}$. Now in general the extension of the product on A to the correct product on A^{**} was achieved by Arens [Are51a; Are51b]. Looking at the formulas, we see that indeed the map (5.4) ends up in the center of the double dual.

That in the situation of Y being equipped with the action of a group H the map (5.4) will be equivariant, is quickly seen. \square

For a discrete group H we recall now the different notions of amenability of H - C^* -algebra from Buss, Echterhoff, and Willett [BEW19, Definitions 2.1 and 4.13]. Note that there are also variants of some of these notions occurring in, e.g., [AD02; BO08]. How these variants are related to each other is explained in [BEW19, Rem. 2.2].

Definition 5.7. Let H be a discrete group and A be an H - C^* -algebra.

- (a) A is called strongly amenable if there is a net $(\theta_i : H \rightarrow Z(\mathcal{M}(A)))_{i \in I}$ of positive type functions²⁸ such that

²⁶Note that we do not need the action to be proper.

²⁷This is just the weak- $*$ topology on $\mathcal{B}(H)$ if we consider it as the dual of $\mathcal{S}^1(H)$, i.e., a net $(T_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{B}(H)$ converges ultra-weakly to T if and only if $(|\text{tr}(ST_\lambda)|)_{\lambda \in \Lambda}$ converges to $|\text{tr}(ST)|$ for every $S \in \mathcal{S}^1(H)$.

²⁸In general, a function $\vartheta : H \rightarrow B$ is of positive type if for any finite subset $\{h_1, \dots, h_n\}$ of H the matrix $(\alpha_{h_i}(\vartheta(h_i^{-1}h_j)))_{i,j} \in M_n(B)$ is positive, where α is the action of H on B [AD87, Def. 2.1].

- each θ_i is finitely supported,
 - for each i we have $\theta_i(e) \leq 1$, and
 - for each $h \in H$ we have $\theta_i(h) \rightarrow 1$ strictly as $i \rightarrow \infty$.
- (b) A is called amenable if there is a net $(\theta_i: H \rightarrow Z(A^{**}))_{i \in I}$ of positive type functions such that
- each θ_i is finitely supported,
 - for each i we have $\theta_i(e) \leq 1$, and
 - for each $h \in H$ we have $\theta_i(h) \rightarrow 1$ ultra-weakly²⁹ as $i \rightarrow \infty$. ♦

Note that strong amenability implies amenability [BEW19, Rem. 2.2].

Proposition 5.8. *Let Y be a proper metric space equipped with an isometric action of a discrete group H .*³⁰

Consider the following three statements:

- (a) *The group H acts amenably on the Higson compactification of Y .*
- (b) *$\bar{c}^{\text{red}}Y$ is an amenable H - C^* -algebra.*
- (c) *We have $\bar{c}^{\text{red}}Y \rtimes_{\max} H \cong \bar{c}^{\text{red}}Y \rtimes_{\text{red}} H$.*

Then we have (a) \Rightarrow (b) \Rightarrow (c) and (a) additionally implies that H is exact.

Proof. We start with the implication (a) \Rightarrow (b) and while doing this we also establish that (a) implies exactness of H .

- We show that (a) implies (b) If H acts amenably on the Higson compactification of Y , then by [AD02, Proposition 6.3] the H - C^* -algebra $C_h(Y)$ is strongly amenable. Since $C_h(Y)$ is commutative, by [BEW19, Lem. 2.5] strong amenability of $C_h(Y)$ in the sense of [AD02] is equivalent to strong amenability of it in the sense of [BEW19, Defn. 2.1]. Therefore there exists a net $(\theta_i: H \rightarrow Z(\mathcal{M}(C_h(Y))))_{i \in I}$ of positive type functions satisfying the corresponding conditions listed in Definition 5.7. But since $C_h(Y)$ is unital and commutative, we have $Z(\mathcal{M}(C_h(Y))) = C_h(Y)$. Composing with (5.4), we get a net $(\theta_i: H \rightarrow Z((\bar{c}^{\text{red}}Y)^{**}))_{i \in I}$ showing that $\bar{c}^{\text{red}}Y$ is amenable.³¹
- Since $C_h(Y)$ is unital, commutative and strongly amenable, this implies that H is exact by [BEW19, Theorem 5.3] (see also [HR00a] for the fact that amenable actions on compact Hausdorff spaces imply exactness).
- The implication (b) \Rightarrow (c) is a completely general fact: amenability of an H - C^* -algebra A implies $A \rtimes_{\max} H \cong A \rtimes_{\text{red}} H$ by [AD87, Proposition 4.8] (see also [BEW19, Section 4]). □

Remark 5.9. Note that $\bar{c}^{\text{red}}Y$ is a strongly amenable H - C^* -algebra if and only if H is amenable.

First, because $\bar{c}^{\text{red}}Y$ is unital, the notions of strong amenability as introduced by Anantharaman-Delaroche [AD02] and by Buss, Echterhoff, and Willett [BEW19] coincide for it [BEW19, Lem. 2.5].

²⁹Recall that a net $(T_\lambda)_{\lambda \in \Lambda}$ in A^{**} converges ultra-weakly to T if and only if $(T_\lambda(\varphi))_{\lambda \in \Lambda}$ converges to $T(\varphi)$ for every $\varphi \in A^*$.

³⁰Note that we do not need here the action of H on Y to be proper.

³¹Here one has to know the fact that if A is a unital C^* -algebra, then the strict topology on $\mathcal{M}(A) = A$ coincides with the norm topology.

If the group H is amenable, then every H - C^* -algebra is strongly amenable. Assume now that $\bar{\mathfrak{c}}^{\text{red}}Y$ is strongly amenable, that is, we have a net $(\theta_i: H \rightarrow Z(\mathcal{M}(\bar{\mathfrak{c}}^{\text{red}}Y)))_{i \in I}$ of positive type functions satisfying the corresponding conditions listed in Definition 5.7. Since $\bar{\mathfrak{c}}^{\text{red}}Y$ is unital we have $\mathcal{M}(\bar{\mathfrak{c}}^{\text{red}}Y) = \bar{\mathfrak{c}}^{\text{red}}Y$, and we further have that $Z(\bar{\mathfrak{c}}^{\text{red}}Y) \cong \mathbb{C}$. Hence the net $(\theta_i)_{i \in I}$ maps actually into \mathbb{C} . But this means that H is amenable. \blacklozenge

Question 5.10. *The above results in combination with the results of [BEW19] suggest that for a proper metric space Y equipped with an isometric action of a discrete group H the following conditions could be equivalent to each other:*

- (a) *The group H acts amenably on the Higson compactification of Y .*
- (b) *$\bar{\mathfrak{c}}^{\text{red}}Y$ is an amenable H - C^* -algebra.*
- (c) *The group H is exact and we have $\bar{\mathfrak{c}}^{\text{red}}Y \rtimes_{\max} H \cong \bar{\mathfrak{c}}^{\text{red}}Y \rtimes_{\text{red}} H$.*

The corresponding version of the above question for the (stable) Higson corona should also be true and is stated in the introduction as Conjecture 1.25.

5.2.2 The case of the (stable) Higson corona

In this section we will adapt the results of the previous one to the (stable) Higson corona. We apply these results to the Gromov boundary of a hyperbolic group in Example 5.13 below.

To treat the case of the (stable) Higson corona, we use the following fact about double duals of quotient C^* -algebras: if $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ is any short exact sequence of C^* -algebras, then we have canonical isomorphisms $(A/I)^{**} \cong A^{**}/I^{**}$ and $A^{**} \cong I^{**} \oplus (A/I)^{**}$ [Bla06, Section III.5.2.11].

The next lemma is an adaption of Lemma 5.6:

Lemma 5.11. *Let Y be a proper metric space.*

Then there is an embedding of $C(\partial_{\text{h}}Y)$ into the center of the double dual of $\mathfrak{c}^{\text{red}}Y$, i.e., an injective and unital $$ -homomorphism*

$$C(\partial_{\text{h}}Y) \rightarrow Z((\mathfrak{c}^{\text{red}}Y)^{**}). \quad (5.5)$$

If Y is further equipped with an isometric action of a discrete group H , then the map (5.5) is equivariant for the induced actions of H on the corresponding C^ -algebras.*

Proof. We have $C(\partial_{\text{h}}Y) \cong C_{\text{h}}(Y)/C_0(Y)$ and $\mathfrak{c}^{\text{red}}Y \cong \bar{\mathfrak{c}}^{\text{red}}Y/C_0(Y, \mathcal{K})$.

The $*$ -homomorphism $f \mapsto f \otimes \text{id}_{\ell^2}$ inducing (5.4) maps the C^* -algebra $C_0(Y)$ to $Z(C_0(Y, \mathcal{K})^{**})$. It extends by [Bla06, Section III.5.2.10] to a normal $*$ -homomorphism³² $C_0(Y)^{**} \rightarrow Z(C_0(Y, \mathcal{K})^{**})$. Considering also the analogous normal extension of (5.4) to $C_{\text{h}}(Y)^{**}$ we conclude that (5.4) induces normal $*$ -homomorphisms mapping the short exact sequence

$$0 \rightarrow C_0(Y)^{**} \rightarrow C_{\text{h}}(Y)^{**} \rightarrow C(\partial_{\text{h}}Y)^{**} \rightarrow 0$$

to the short exact sequence

$$0 \rightarrow C_0(Y, \mathcal{K})^{**} \rightarrow (\bar{\mathfrak{c}}^{\text{red}}Y)^{**} \rightarrow (\mathfrak{c}^{\text{red}}Y)^{**} \rightarrow 0$$

³²A normal $*$ -homomorphism is one which is continuous for the respective ultra-weak topologies [Bla06, Section III.2.2].

and the image of $C(\partial_h Y)^{**}$ will be contained in $Z((\mathfrak{c}^{\text{red}}Y)^{**})$. The restriction of the map on $C(\partial_h Y)^{**}$ to its C^* -subalgebra $C(\partial_h Y)$ is the sought map (5.5).

Let us show injectivity of (5.5). General theory [Bla06, Section III.5.2.11] tells us that there is a central projection $p \in (\bar{\mathfrak{c}}^{\text{red}}Y)^{**}$ such that $p \cdot (\bar{\mathfrak{c}}^{\text{red}}Y)^{**}$ is $C_0(Y, \mathcal{K})^{**}$ and such that $(1 - p) \cdot (\bar{\mathfrak{c}}^{\text{red}}Y)^{**}$ is $(\mathfrak{c}^{\text{red}}Y)^{**}$. This central projection is given as the supremum of an approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ in $C_0(Y, \mathcal{K})$. Let $f \in C_h(Y)$ be non-zero in $C(\partial_h Y)$. We have to show that it is still non-zero in $(\mathfrak{c}^{\text{red}}Y)^{**}$, i.e., that it does not lie in $C_0(Y, \mathcal{K})^{**}$. Because f is non-zero in $C(\partial_h Y)$, there exists a point $x \in \partial_h Y$ such that $\text{ev}_x(f) \neq 0$. We choose any unit vector v in the Hilbert space H (the auxiliary Hilbert space used in the definition of $\bar{\mathfrak{c}}^{\text{red}}Y$) and define a linear functional η on $\bar{\mathfrak{c}}^{\text{red}}Y$ by

$$\eta(g) := \text{ev}_x(g_v),$$

where $g_v \in C_h(Y)$ is the function given by $g_v(y) := \varphi_v(g(y))$ using the vector state $\varphi_v(T) := \langle Tv, v \rangle$ on H . Because $\eta: \bar{\mathfrak{c}}^{\text{red}}Y \rightarrow \mathbb{C}$ is a positive linear map, it extends to a normal positive linear map $\eta^{**}: (\bar{\mathfrak{c}}^{\text{red}}Y)^{**} \rightarrow \mathbb{C}^{**} \cong \mathbb{C}$ and we have $\eta^{**}(f \otimes \text{id}_{\ell^2}) = \text{ev}_x(f) \neq 0$. For $\theta \in C_0(Y, \mathcal{K})^{**}$ we have $\eta^{**}(\theta) = 0$ since (choosing $(\theta_\mu)_{\mu \in \Lambda'} \in C_0(Y, \mathcal{K})$ approximating θ ultra-weakly)

$$\begin{aligned} \eta^{**}(\theta) &= \eta^{**}(p \cdot \theta) \\ &= \eta^{**}\left(\lim_{\lambda \rightarrow \infty} u_\lambda \cdot \theta\right) \\ &= \lim_{\lambda \rightarrow \infty} \eta^{**}(u_\lambda \cdot \theta) \\ &= \lim_{\lambda \rightarrow \infty} \eta^{**}\left(u_\lambda \cdot \lim_{\mu \rightarrow \infty} \theta_\mu\right) \\ &= \lim_{\lambda \rightarrow \infty} \lim_{\mu \rightarrow \infty} \eta^{**}(u_\lambda \cdot \theta_\mu) \end{aligned}$$

(where we used that η^{**} is ultra-weakly continuous and multiplication separately ultra-weakly continuous) and $\eta^{**}(u_\lambda \cdot \theta_\mu) = 0$ since $u_\lambda \cdot \theta_\mu \in C_0(Y, \mathcal{K})$. Because $\eta^{**}(f \otimes \text{id}_{\ell^2}) \neq 0$, this means that $f \notin C_0(Y, \mathcal{K})^{**}$ finishing the proof that (5.5) is injective.

The other statements about the map (5.5) are straight-forward to prove, which finishes this proof. \square

Proposition 5.12. *Let Y be a proper metric space equipped with an isometric action of a discrete group H .*

Consider the following three statements:

- (a) *The group H acts amenably on the Higson corona of Y .*
- (b) *$\mathfrak{c}^{\text{red}}Y$ is an amenable H - C^* -algebra.*
- (c) *We have $\mathfrak{c}^{\text{red}}Y \rtimes_{\max} H \cong \mathfrak{c}^{\text{red}}Y \rtimes_{\text{red}} H$.*

Then we have (a) \Rightarrow (b) \Rightarrow (c) and (a) additionally implies that H is exact.

Proof. Completely analogous to the proof of Proposition 5.8. \square

Example 5.13. Let H be a Gromov hyperbolic group.

It is known that in this case H acts amenably on its Gromov boundary [AD02, Example 2.7.4]. Since the Gromov boundary is dominated by the Higson corona, i.e., there is a natural H -map $\partial_h H \rightarrow \partial_{\text{Gromov}} H$, it follows that H also acts amenably on its Higson corona.

Proposition 5.12 then implies that $\mathfrak{c}^{\text{red}}H$ is an amenable H - C^* -algebra and $\mathfrak{c}^{\text{red}}H \rtimes_{\max} H \cong \mathfrak{c}^{\text{red}}H \rtimes_{\text{red}} H$. \blacklozenge

5.3 Construction of the equivariant slant products

Just like in Section 3.2 let us fix an ample X - G -module (H_X, ρ_X, u_G) and an ample Y - H -module (H_Y, ρ_Y, u_H) .

5.3.1 The slant product on the equivariant Roe algebra

We denote by $\rho'_Y: C_0(Y, \mathcal{K}) \rightarrow \mathcal{B}(H_Y \otimes \ell^2)$ the tensor product of the given representation ρ_Y and the canonical representation of \mathcal{K} on ℓ^2 . The H -action on Y induces an H -action on $C_0(Y, \mathcal{K})$ and $(\rho'_Y, u_H \otimes \text{id}_{\ell^2})$ becomes a covariant pair. Because every automorphism of a C^* -algebra extends uniquely to an automorphism of its multiplier algebra, we get a covariant pair $(\bar{\rho}_Y, u_H \otimes \text{id}_{\ell^2})$ for $(\mathcal{M}(C_0(Y, \mathcal{K}), H))$. Here $\bar{\rho}_Y$ is precisely the same as in (4.2). We amplify the latter covariant pair via the left-regular representation $\lambda_H: H \rightarrow \mathcal{U}(\ell^2(H))$ to obtain a covariant pair $(\hat{\rho}_Y, \hat{u}_H)$, where $\hat{\rho}_Y := \text{id}_{\ell^2(H)} \otimes \bar{\rho}_Y$ and $\hat{u}_H := \lambda_H \otimes u_H \otimes \text{id}_{\ell^2}$.

We consider $\bar{\mathfrak{c}}^{\text{red}}Y \subset \mathcal{B}(\ell^2(H) \otimes H_Y \otimes \ell^2)$ via the representation $\hat{\rho}_Y$. It follows from the above that the H - C^* -algebra $\bar{\mathfrak{c}}^{\text{red}}Y$ is covariantly represented on $\ell^2(H) \otimes H_Y \otimes \ell^2$. Moreover, by Fell's absorption principle [BO08, Proposition 4.1.7] this yields an embedding of the reduced crossed product $\hat{\rho}_Y \rtimes \hat{u}_H: \bar{\mathfrak{c}}^{\text{red}}Y \rtimes_{\text{red}} H \hookrightarrow \mathcal{B}(\ell^2(H) \otimes H_Y \otimes \ell^2)$.

Let us redefine $H_{X \times Y} := H_X \otimes \ell^2(H) \otimes H_Y$. The Hilbert space $H_{X \times Y}$ has a unitary representation $u_{G \times H}$ of $G \times H$ via $g, h \mapsto u_G(g) \otimes \lambda_H(h) \otimes u_H(h)$ and a representation $\rho_{X \times Y}$ of $C_0(X \times Y)$ via $f \otimes f' \mapsto \rho_X(f) \otimes \text{id}_{\ell^2(H)} \otimes \rho_Y(f')$. This turns $H_{X \times Y}$ into an ample $(X \times Y)$ - $(G \times H)$ -module. Similarly as in (4.1), we define $\tilde{H}_X := H_{X \times Y} \otimes \ell^2 = H_X \otimes \ell^2(H) \otimes H_Y \otimes \ell^2$ and

$$\tilde{\rho}_X := \rho_X \otimes \text{id}_{\ell^2(H) \otimes H_Y \otimes \ell^2}: C_0(X) \rightarrow \mathcal{B}(\tilde{H}_X).$$

In addition, let $\tilde{\rho}_Y := \text{id}_{H_X} \otimes \hat{\rho}_Y$ and $\tilde{u}_H := \text{id}_{H_X} \otimes \hat{u}_H$. Then we obtain $\tilde{\rho}_Y \rtimes \tilde{u}_H = \text{id}_{H_X} \otimes (\hat{\rho}_Y \rtimes \hat{u}_H): \bar{\mathfrak{c}}^{\text{red}}Y \rtimes_{\text{red}} H \hookrightarrow \mathcal{B}(\tilde{H}_X)$. Note that the definitions of $\tilde{\rho}_Y$ and $\tilde{\rho}_X$ are slightly different than in Section 4.1.1 because here we use $\ell^2(H)$ as an additional tensor factor in \tilde{H}_X . This allows us to use the reduced crossed product.

Now the following equivariant versions of the lemmas from Section 4.1.1 hold. Here we let $E_G^*(\tilde{\rho}_X) \subset \mathcal{B}(\tilde{H}_X)$ denote the C^* -algebra generated by all the G -equivariant operators of finite propagation. Then by the same argument as for Lemma 4.2, the C^* -algebra $C_G^*(\tilde{\rho}_X)$ is an ideal in $E_G^*(\tilde{\rho}_X)$.

Lemma 5.14. *The images of the two representations*

$$\begin{aligned} \tau: C_{G \times H}^*(\rho_{X \times Y}) &\rightarrow \mathcal{B}(\tilde{H}_X) \text{ given by } S \mapsto S \otimes \text{id}_{\ell^2}, \\ \tilde{\rho}_Y \rtimes \tilde{u}_H: \bar{\mathfrak{c}}^{\text{red}}Y \rtimes_{\text{red}} H &\rightarrow \mathcal{B}(\tilde{H}_X) \text{ defined above,} \end{aligned}$$

are contained in $E_G^*(\tilde{\rho}_X)$.

Proof. The operators considered here are clearly G -equivariant. Thus for the first part the argument is the same as for Lemma 4.3. For the second part, we use in addition that the operators $\tilde{u}_H(h)$ for $h \in H$ commute with the image of $\tilde{\rho}_X$ and hence have propagation zero. \square

Lemma 5.15. *The images of the $*$ -homomorphisms $C_{G \times H}^*(\rho_{X \times Y}) \rightarrow E_G^*(\tilde{\rho}_X)$ and $\bar{\mathfrak{c}}^{\text{red}}Y \rtimes_{\text{red}} H \rightarrow E_G^*(\tilde{\rho}_X)$ obtained from the previous lemma commute up to $C_G^*(\tilde{\rho}_X)$.*

Proof. The image of $\tilde{\rho}_Y \rtimes \tilde{u}_H: \bar{\mathfrak{c}}^{\text{red}}Y \rtimes_{\text{red}} H \rightarrow E_G^*(\tilde{\rho}_X)$ has a dense subset consisting of linear combinations of products of operators of the form $\tilde{\rho}_Y(f) = \text{id}_{H_X} \otimes \text{id}_{\ell^2(H)} \otimes$

$\bar{\rho}_Y(f)$, where $f \in \bar{\mathfrak{c}}^{\text{red}}(f)$, with operators of the form $\tilde{u}_H(h) = \text{id}_{H_X} \otimes \lambda_H(h) \otimes u_H(h) \otimes \text{id}_{\ell^2}$, where $h \in H$. Thus it suffices to show that the commutators

$$[S \otimes \text{id}_{\ell^2}, \text{id}_{H_X} \otimes \text{id}_{\ell^2(H)} \otimes \bar{\rho}_Y(f)], \quad (5.6)$$

$$[S \otimes \text{id}_{\ell^2}, \tilde{u}_H(h)] \quad (5.7)$$

are contained in $C_G^*(\tilde{\rho}_X)$ for $S \in C_{G \times H}^*(\rho_{X \times Y})$, $f \in \bar{\mathfrak{c}}^{\text{red}}(Y)$ and $h \in H$. We may also assume that S has finite propagation. Then the proof of Lemma 4.5 (applied to the Y -module $\ell^2(H) \otimes H_Y$ with the representation $\text{id}_{\ell^2(H)} \otimes \rho_Y$) shows that (5.6) is an element of $C^*(\tilde{\rho}_X)$ with finite propagation. It is also G -equivariant because S is. Hence (5.6) is an element of $C_G^*(\tilde{\rho}_X)$. Finally, $S \otimes \text{id}_{\ell^2}$ is H -equivariant because S is. That is, it commutes with $\tilde{u}_H(h)$ and thus (5.7) is zero. \square

Hence as an analogue of (4.3) we get an induced $*$ -homomorphism

$$\Phi: C_{G \times H}^*(\rho_{X \times Y}) \otimes_{\max} (\bar{\mathfrak{c}}^{\text{red}} Y \rtimes_{\text{red}} H) \rightarrow E_G^*(\tilde{\rho}_X)/C_G^*(\tilde{\rho}_X)$$

given by $S \otimes f \delta_h \mapsto [\tau(S) \circ \tilde{\rho}_Y(f) \circ \tilde{u}_H(h)]$.

A slight elaboration of the proof of Lemma 4.6 shows the following.

Lemma 5.16. *The above $*$ -homomorphism Φ factors through the C^* -algebra*

$$C_{G \times H}^*(\rho_{X \times Y}) \otimes_{\max} \frac{\bar{\mathfrak{c}}^{\text{red}} Y \rtimes_{\text{red}} H}{C_0(Y, \mathcal{K}) \rtimes_{\text{red}} H}.$$

That is, it defines a $$ -homomorphism*

$$C_{G \times H}^*(\rho_{X \times Y}) \otimes_{\max} \frac{\bar{\mathfrak{c}}^{\text{red}} Y \rtimes_{\text{red}} H}{C_0(Y, \mathcal{K}) \rtimes_{\text{red}} H} \rightarrow E_G^*(\tilde{\rho}_X)/C_G^*(\tilde{\rho}_X). \quad (5.8)$$

Unfortunately, in general we cannot use $\bar{\mathfrak{c}}^{\text{red}} Y \rtimes_{\text{red}} H$ in the above due to a potential failure of exactness in the sequence $C_0(Y, \mathcal{K}) \rtimes_{\text{red}} H \rightarrow \bar{\mathfrak{c}}^{\text{red}}(Y) \rtimes_{\text{red}} H \rightarrow \mathfrak{c}^{\text{red}}(Y) \rtimes_{\text{red}} H$. If H is an exact group, then this presents no issues. More generally, we can remedy this by using an exact crossed functor \rtimes_{μ} in the sense of [BGW16, Definition 3.1] instead of \rtimes_{red} . Indeed, we deduce the next proposition immediately from (5.8).

Proposition 5.17. *Let \rtimes_{μ} be an exact crossed product functor. Then there is a homomorphism*

$$\Psi_{\mu}: C_{G \times H}^*(\rho_{X \times Y}) \otimes_{\max} (\mathfrak{c}^{\text{red}} Y \rtimes_{\mu} H) \rightarrow E_G^*(\tilde{\rho}_X)/C_G^*(\tilde{\rho}_X) \quad (5.9)$$

induced by $S \otimes [f] \delta_h \mapsto [\tau(S) \circ \tilde{\rho}_Y(f) \circ \tilde{u}_H(h)]$.

Definition 5.18. By the same construction as in Definition 4.7, we obtain the slant product

$$K_p(C_{G \times H}^*(X \times Y)) \otimes K_{1-q}(\mathfrak{c}^{\text{red}} Y \rtimes_{\mu} H) \rightarrow K_{p-q}(C_G^* X) \quad (5.10)$$

for any exact crossed product functor \rtimes_{μ} . If H is an exact group, we also obtain (5.10) for $\mu = \text{red}$. \blacklozenge

5.3.2 The slant product on the equivariant localization algebras

We define $E_{G,L}^*(\tilde{\rho}_X)$ as the C^* -subalgebra of $C_b([1, \infty), E_G^*(\tilde{\rho}_X))$ generated by the bounded and uniformly continuous functions $S: [1, \infty) \rightarrow E_G^*(\tilde{\rho}_X)$ such that the propagation of $S(t)$ is finite for all $t \geq 1$ and tends to zero as $t \rightarrow \infty$. Similarly we define $E_{G,L,0}^*(\tilde{\rho}_X)$ as the ideal in $E_{G,L}^*(\tilde{\rho}_X)$ consisting of all maps that vanish at 1. Note that $C_{G,L}^*(\tilde{\rho}_X)$ is an ideal in $E_{G,L}^*(\tilde{\rho}_X)$ and $C_{G,L,0}^*(\tilde{\rho}_X)$ is even an ideal in all of the three $E_{G,L}^*(\tilde{\rho}_X)$, $E_{G,L,0}^*(\tilde{\rho}_X)$ and of course $C_{G,L}^*(\tilde{\rho}_X)$.

Lemma 5.19. *The following analogue of Lemma 4.8 holds.*

(i) *The images of the two isometric *-homomorphisms*

$$\tau_L: C_{G \times H, L}^*(\rho_{X \times Y}) \rightarrow C_b([1, \infty), \mathcal{B}(\tilde{H}_X)),$$

which is obtained by applying the functor $C_b([1, \infty), -)$ to τ , and

$$\tilde{\rho}_{Y, H, L}: \bar{c}^{\text{red}}Y \rtimes_{\text{red}} H \xrightarrow{\tilde{\rho}_Y \rtimes_{\text{red}} \tilde{u}_H} \mathcal{B}(\tilde{H}_X) \xrightarrow[\text{as constant functions}]{\text{inclusion}} C_b([1, \infty), \mathcal{B}(\tilde{H}_X)),$$

are contained in $E_{G, L}^(\tilde{\rho}_X)$.*

(ii) *The image of τ_L commutes up to $C_{G, L}^*(\tilde{\rho}_X)$ with the image of $\bar{c}^{\text{red}}Y \rtimes_{\text{red}} H$ under $\tilde{\rho}_{Y, H, L}$ and the image of $C_{L, 0}^*(\rho_{X \times Y})$ under τ_L commutes up to $C_{L, 0}^*(\tilde{\rho}_X)$ with the image of $\bar{c}^{\text{red}}Y$ under $\tilde{\rho}_{Y, L}$. Hence they induce *-homomorphisms*

$$\begin{aligned} \Phi_L: C_{G \times H, L}^*(\rho_{X \times Y}) \otimes_{\max} \bar{c}^{\text{red}}Y \rtimes_{\text{red}} H &\rightarrow E_{G, L}^*(\tilde{\rho}_X) / C_{G, L}^*(\tilde{\rho}_X) \\ \Phi_{L, 0}: C_{G \times H, L, 0}^*(\rho_{X \times Y}) \otimes_{\max} \bar{c}^{\text{red}}Y \rtimes_{\text{red}} H &\rightarrow E_{G, L}^*(\tilde{\rho}_X) / C_{G, L, 0}^*(\tilde{\rho}_X) \end{aligned}$$

given by $S \otimes f \mapsto [\tau_L(S) \circ \tilde{\rho}_{Y, H, L}(f)]$ and the image of $\Phi_{L, 0}$ is even contained in $E_{G, L, 0}^(\tilde{\rho}_X) / C_{G, L, 0}^*(\tilde{\rho}_X)$.*

(iii) *Let μ be an exact crossed product functor. Then the *-homomorphisms Φ_L and $\Phi_{L, 0}$ factor through $C_L^*(\rho_{X \times Y}) \otimes_{\max} \bar{c}^{\text{red}}Y \rtimes_{\mu} H$ and $C_{L, 0}^*(\rho_{X \times Y}) \otimes_{\max} \bar{c}^{\text{red}}Y \rtimes_{\mu} H$, respectively. In other words, they define *-homomorphisms*

$$\begin{aligned} \Psi_L: C_{G \times H, L}^*(\rho_{X \times Y}) \otimes_{\max} \bar{c}^{\text{red}}Y \rtimes_{\mu} H &\rightarrow E_{G, L}^*(\tilde{\rho}_X) / C_{G, L}^*(\tilde{\rho}_X), \\ \Psi_{L, 0}: C_{G \times H, L, 0}^*(\rho_{X \times Y}) \otimes_{\max} \bar{c}^{\text{red}}Y \rtimes_{\mu} H &\rightarrow E_{G, L, 0}^*(\tilde{\rho}_X) / C_{G, L, 0}^*(\tilde{\rho}_X). \end{aligned}$$

Definition 5.20. Let μ be an exact crossed product functor. By Lemma 5.19 we obtain the following slant products analogously as in Definition 4.9.

$$\begin{aligned} K_p^{G \times H}(X \times Y) \otimes K_{1-q}(\bar{c}^{\text{red}}Y \rtimes_{\mu} H) &\rightarrow K_{p-q}^G(X), \\ S_p^{G \times H}(X \times Y) \otimes K_{1-q}(\bar{c}^{\text{red}}Y \rtimes_{\mu} H) &\rightarrow S_{p-q}^G(X). \end{aligned}$$

If the group H is exact, we also obtain the above for $\mu = \text{red}$. ♦

5.4 Properties of the equivariant slant products

In this section we state the properties of the equivariant slant products which are analogous to those of the non-equivariant one.

The following compatibility of the slant products with the equivariant Higson–Roe sequence is proven in exactly the same way as its non-equivariant counterpart Theorem 4.10 by decorating everything with the groups G and H .

Theorem 5.21. *The diagram*

$$\begin{array}{ccccccc} S_p^{G \times H}(X \times Y) & \longrightarrow & K_p^{G \times H}(X \times Y) & \longrightarrow & K_p(C_{G \times H}^*(X \times Y)) & \xrightarrow{\partial} & S_{p-1}^{G \times H}(X \times Y) \\ \downarrow / \theta & & \downarrow / \theta & & \downarrow / \theta & & \downarrow / \theta \\ S_{p-q}^G(X) & \longrightarrow & K_{p-q}^G(X) & \longrightarrow & K_{p-q}(C_G^*X) & \xrightarrow{\partial} & K_{p-1-q}^G(X) \end{array}$$

commutes for every $\theta \in K_{1-q}(\bar{c}^{\text{red}}Y \rtimes_{\mu} H)$. □

In Section 4.3 we had shown that our non-equivariant slant product for the localization algebra factors through co-assembly and the usual slant product between K-homology and K-theory. Now we shall show that our slant product for the equivariant localization algebra also factors through our version of co-assembly from Section 5.1. The first part of Section 4.3 goes through equivariantly. That is, one obtains an isomorphism

$$K_*(C_{G,L}^*(X)) \cong E_*^G(C_0(X), \mathbb{C})$$

and the latter group is, by definition, the equivariant K-homology group $K_*^G(X)$ in the E-theory picture. However, there is no already well known slant product between equivariant K-homology and equivariant K-theory defined via equivariant E-theory, as explained earlier. Instead we turn an equivariant version of Lemma 4.19 into a definition.

Consider the map

$$\begin{aligned} \Upsilon_L: C_{G \times H, L}^*(\rho_{X \times Y}) \otimes C_0(Y, \mathcal{K}) \rtimes H &\rightarrow C_{G, L}^*(\tilde{\rho}_X)/C_0([1, \infty), C_G^*(\tilde{\rho}_X)), \\ (T_t)_{t \in [1, \infty)} \otimes f &\mapsto [(\tau(T_t) \circ (\tilde{\rho}_Y \times \tilde{u}_H)(f))_{t \in [1, \infty)}]. \end{aligned}$$

By enriching the reasoning in Section 4.3 with the ideas of the proofs of Lemmas 5.14 and 5.15 we get that Υ_L is a well-defined *-homomorphism.

Definition 5.22. The slant product between equivariant K-homology and equivariant K-theory is defined as the composition

$$\begin{aligned} K_p^{G \times H}(X \times Y) \otimes K_H^q(Y) &\cong K_p(C_{G \times H, L}^*(\rho_{X \times Y})) \otimes K_{-q}(C_0(Y, \mathcal{K}) \rtimes H) \\ &\xrightarrow{\boxtimes} K_{p-q}(C_{G \times H, L}^*(\rho_{X \times Y}) \otimes C_0(Y, \mathcal{K}) \rtimes H) \\ &\xrightarrow{(\Upsilon_L)^*} K_{p-q}(C_{G, L}^*(\tilde{\rho}_X)/C_0([1, \infty), C_G^*(\tilde{\rho}_X))) \\ &\xrightarrow{\cong} K_{p-q}(C_{G, L}^*(\tilde{\rho}_X)) \cong K_{p-q}^G(X), \end{aligned}$$

where the third map is the inverse of the isomorphism induced on K-theory by the canonical projection $C_{G, L}^*(\tilde{\rho}_X) \rightarrow C_{G, L}^*(\tilde{\rho}_X)/C_0([1, \infty), C_G^*(\tilde{\rho}_X))$. \blacklozenge

The proofs of Proposition 4.17, Corollary 4.18 and Theorem 4.13 in Section 4.3 now generalize to the equivariant case to yield the following equivariant version of Theorem 4.13.

Theorem 5.23. *Let Y have continuously bounded geometry. The slant product on the localization algebra from Definition 5.20 and the slant product from Definition 5.22 are related to each other via the co-assembly map*

$$\mu_H^*: K_*(\mathfrak{c}^{\text{red}}Y \rtimes_\mu H) \rightarrow K_H^{1-*}(Y)$$

in the sense that $x/\theta = x/\mu_H^*\theta$ for all $x \in K_*^{G \times H}(X \times Y)$ and $\theta \in K_*(\mathfrak{c}^{\text{red}}Y \rtimes_\mu H)$.

Next, Theorem 4.20 and Corollary 4.23 can also be generalized to the equivariant cases.

Theorem 5.24. *Let G, H, K be countable discrete groups acting properly and isometrically on proper metric spaces X, Y, Z and assume that Z has bounded geometry. Then the compositions*

$$\begin{aligned} K_m^G(X) &\xrightarrow{\times z} K_{m+p}^{G \times H \times K}(X \times Y \times Z) \xrightarrow{/\theta} K_{m+p-q}^{G \times H}(X \times Y) \\ S_m^G(X) &\xrightarrow{\times z} S_{m+p}^{G \times H \times K}(X \times Y \times Z) \xrightarrow{/\theta} S_{m+p-q}^{G \times H}(X \times Y) \\ K_m(C_G^*X) &\xrightarrow{\times z} K_{m+p}(C_{G \times H \times K}^*(X \times Y \times Z)) \xrightarrow{/\theta} K_{m+p-q}(C_{G \times H}^*(X \times Y)) \end{aligned}$$

are equal to the external product with the appropriate slant product z/θ for all $m, p, q \in \mathbb{Z}$ and all z, θ as follows:

- In the first two compositions $z \in K_p^{H \times K}(Y \times Z)$ and in the third one $z \in K_p(C_{H \times K}^*(Y \times Z))$.
- In the first composition either $\theta \in K_K^q(Z)$ or $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Z \rtimes_\mu K)$ and in the other two $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Z \rtimes_\mu K)$. \square

Outline of proof. The proof works almost word by word the same, just that one has to replace the Hilbert space H_Z by $\ell^2(K) \otimes H_Z$ at several places within the constructions. The only difference is the case $\theta \in K_K^q(Z)$, because we didn't define the equivariant version of this slant product via E -theory and hence it does not follow from abstract properties of E -theory. Instead one has to prove it along the lines of the other cases, but using the map Υ_L instead of Ψ, Ψ_L or $\Psi_{L,0}$. \square

Definition 5.25. The pairing

$$\langle -, - \rangle: K_p(C_H^*Y) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y \rtimes_\mu H) \rightarrow K_{p-q}(C^*\{*\}) \cong \begin{cases} \mathbb{Z} & p - q \text{ even} \\ 0 & p - q \text{ odd} \end{cases}$$

is defined as the special case of the equivariant slant product where $X = \{*\}$ is a single point equipped with the action of the trivial group. The same construction applied to the localization algebra instead of the Roe algebra also yields a pairing

$$\langle -, - \rangle: K_p^H(Y) \otimes K_{1-q}(\mathfrak{c}^{\text{red}}Y \rtimes_\mu H) \rightarrow K_{p-q}(\{*\}) \cong \begin{cases} \mathbb{Z} & p - q \text{ even} \\ 0 & p - q \text{ odd} \end{cases}$$

and similarly we also have a pairing

$$\langle -, - \rangle: K_p^H(Y) \otimes K_H^q(Y) \rightarrow K_{p-q}(\{*\}) \cong \begin{cases} \mathbb{Z} & p - q \text{ even} \\ 0 & p - q \text{ odd.} \end{cases} \quad \blacklozenge$$

Corollary 5.26. *The compositions*

$$\begin{array}{ccccc} K_m^G(X) & \xrightarrow{\times z} & K_{m+p}^{G \times H}(X \times Y) & \xrightarrow{/\theta} & K_{m+p-q}^G(X) \\ S_m^G(X) & \xrightarrow{\times z} & S_{m+p}^{G \times H}(X \times Y) & \xrightarrow{/\theta} & S_{m+p-q}^G(X) \\ K_m(C_G^*X) & \xrightarrow{\times z} & K_{m+p}(C_{G \times H}^*(X \times Y)) & \xrightarrow{/\theta} & K_{m+p-q}(C_G^*X) \end{array}$$

are equal to the multiplication with $\langle z, \theta \rangle$, which is either an integer, if $p - q$ is even, or zero by construction, if $p - q$ odd, for all $m, p, q \in \mathbb{Z}$ and all z, θ as follows:

- In the first two compositions $z \in K_p^H(Y)$ and in the third $z \in K_p(C_H^*Y)$.
- In the first composition either $\theta \in K_H^q(Y)$ or $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Y \rtimes_\mu H)$ and in the other two $\theta \in K_{1-q}(\mathfrak{c}^{\text{red}}Y \rtimes_\mu H)$. \square

Next on the list is naturality.

Definition 5.27. Let X, X' be proper metric spaces equipped with proper isometric actions by the same countable discrete group G and fix an ample X - G -module (H_X, ρ_X, u_G) and an ample X' - G -module $(H_{X'}, \rho_{X'}, u'_G)$. An isometry $V: H_X \rightarrow H_{X'}$ is said to *equivariantly cover* an equivariant coarse map $\alpha: X \rightarrow X'$ if it covers α in the sense of Definition 4.25 and is in addition equivariant with respect to u_G and u'_G . Similarly, a uniformly continuous family of isometries $V: [1, \infty) \rightarrow \mathcal{B}(H_X, H_{X'})$ is said to *equivariantly cover* an equivariant uniformly continuous coarse map $\alpha: X \rightarrow X'$ if it covers α in the sense of Definition 4.29 and is in addition equivariant. \blacklozenge

Proposition 5.28 (see [WY19, Proposition 4.5.11, Theorems 5.2.6 and 6.6.3]). *Equivariantly covering (uniformly continuous families of) isometries as in the previous definition always exist. Conjugation with an isometry V which equivariantly covers a coarse map α yields a $*$ -homomorphism*

$$\text{Ad}_V: C_G^*(\rho_X) \rightarrow C_G^*(\rho_{X'})$$

and conjugation with a uniformly continuous family of isometries V which equivariantly covers a uniformly continuous equivariant coarse map α yields $*$ -homomorphisms

$$\text{Ad}_V: C_{G,L}^*(\rho_X) \rightarrow C_{G,L}^*(\rho_{X'}), \quad \text{Ad}_V: C_{G,L,0}^*(\rho_X) \rightarrow C_{G,L,0}^*(\rho_{X'}).$$

The induced maps on K-theory

$$K_*(C_G^*X) \rightarrow K_*(C_G^*X'), \quad K_*^G(X) \rightarrow K_*^G(X'), \quad S_*^G(X) \rightarrow S_*^G(X')$$

are independent of all choices, depend functorial on α and will all be denoted by α_* . Furthermore, they make the diagram

$$\begin{array}{ccccccc} K_{*+1}(C_G^*X) & \xrightarrow{\partial} & S_*^G(X) & \longrightarrow & K_*^G(X) & \xrightarrow{\text{Ind}} & K_*(C_G^*X) \\ \downarrow \alpha_* & & \downarrow \alpha_* & & \downarrow \alpha_* & & \downarrow \alpha_* \\ K_{*+1}(C_G^*X') & \xrightarrow{\partial} & S_*^G(X') & \longrightarrow & K_*^G(X') & \xrightarrow{\text{Ind}} & K_*(C_G^*X') \end{array}$$

commute.

Again, functoriality of $K_*^G(-)$ can be extended to G -equivariant proper continuous maps and functoriality of $S_*^G(-)$ can be extended to G -equivariant continuous coarse maps by using different pictures of these groups. This functoriality can be dealt with in exactly the same way as in the non-equivariant case as described in Remark 4.32.

On the other side, contravariant functoriality of $K_*(\mathfrak{c}^{\text{red}}(-) \rtimes H)$ under H -equivariant coarse maps and of $K_H^*(-)$ under H -equivariant proper continuous maps is clear.

The equivariance can be implemented very easily into the proofs of Theorem 4.28 and Theorem 4.31 and the proof can also be adapted to the slant product between equivariant K-homology and equivariant K-theory defined in Definition 5.22 by using the map Υ_L instead of Ψ_L . One obtains the following.

Theorem 5.29. *All the equivariant slant products are natural in the sense that the formula*

$$\alpha_*(x/\beta^*(\theta)) = (\alpha \times \beta)_*(x)/\theta$$

holds in each of the following cases:

- $\alpha: X \rightarrow X'$ and $\beta: Y \rightarrow Y'$ are equivariant coarse maps, $x \in K_*(C_{G \times H}^*(X \times Y))$ and $\theta \in K_*(\mathfrak{c}^{\text{red}}Y' \rtimes_\mu H)$.
- $\alpha: X \rightarrow X'$ is an equivariant proper continuous map and $\beta: Y \rightarrow Y'$ is a equivariant continuous coarse maps, $x \in K_*^{G \times H}(X \times Y)$ and $\theta \in K_*(\mathfrak{c}^{\text{red}}Y' \rtimes_\mu H)$.
- $\alpha: X \rightarrow X'$ and $\beta: Y \rightarrow Y'$ are equivariant continuous coarse maps, $x \in S_*^{G \times H}(X \times Y)$ and $\theta \in K_*(\mathfrak{c}^{\text{red}}Y' \rtimes_\mu H)$.
- $\alpha: X \rightarrow X'$ and $\beta: Y \rightarrow Y'$ are equivariant proper continuous maps, $x \in K_*^{G \times H}(X \times Y)$ and $\theta \in K_H^*(Y')$. \square

Finally, using equivariant functoriality we get the same coarsification results as in Section 4.6. More precisely, given any proper metric space X equipped with a proper isometric action by a countable discrete group G , it is possible to choose the discrete coarsely equivalent subspace $X' \subset X$ to be G -invariant. Then each Rips complex $P_R X'$ inherits a canonical proper isometric G -action such that all the inclusion maps $P_R X' \rightarrow P_S X'$ for $R \leq S$ are equivariant continuous coarse equivalences.

Then we can define the equivariant coarse K-homology and the equivariant coarse structure group as

$$\mathrm{KX}_*^G(X) := \varinjlim_{R \geq 0} \mathrm{K}_*^G(P_R X') \quad \text{and} \quad \mathrm{SX}_*^G(X) := \varinjlim_{R \geq 0} \mathrm{S}_*^G(P_R X')$$

and similarly there is also a equivariant coarse K-theory which fits into a Milnor- \varprojlim^1 -sequence

$$0 \rightarrow \varprojlim_{R \geq 0} {}^1\mathrm{K}_G^{*+1}(P_R X') \rightarrow \mathrm{KX}_*^G(X) \rightarrow \varprojlim_{R \geq 0} \mathrm{K}_G^*(P_R X') \rightarrow 0.$$

All of the above groups are functorial under equivariant coarse maps in the obvious way and the Milnor- \varprojlim^1 -sequence is natural.

Again, the construction does not yield anything new for the $\mathrm{K}_*(C_G^* X)$ and $\mathrm{K}_*(\mathfrak{c}^{\mathrm{red}} X \rtimes_\mu G)$. There is a natural equivariant coarsified version of the Higson–Roe sequence

$$\cdots \rightarrow \mathrm{K}_{*+1}(C_G^* X) \rightarrow \mathrm{SX}_*^G(X) \rightarrow \mathrm{KX}_*^G(X) \xrightarrow{\mu_G} \mathrm{K}_*(C_G^* X) \rightarrow \cdots$$

and also a natural equivariant coarse co-assembly map

$$\mu_G^*: \mathrm{K}_{1-*}(\mathfrak{c}^{\mathrm{red}} X \rtimes_\mu G) \rightarrow \mathrm{KX}_*^G(X).$$

Furthermore, the map $X \rightarrow P_{2R} X'$ defined in (4.14) can also be assumed to be equivariant by using an equivariant partition of unity $\{\varphi_{x'}\}_{x' \in X}$, i. e. one for which $\varphi_{gx'}(gx) = \varphi_{x'}(x)$ for all $x \in X$, $x' \in X'$ and $g \in G$, and it is unique up to equivariant homotopy equivalence which stays close to the identity. Therefore we have canonical equivariant coarsification maps

$$c_G: \mathrm{K}_*^G(X) \rightarrow \mathrm{KX}_*^G(X) \quad \text{and} \quad c_G: \mathrm{S}_*^G(X) \rightarrow \mathrm{SX}_*^G(X)$$

and a canonical equivariant co-coarsification map

$$c_G^*: \mathrm{KX}_*^G(X) \rightarrow \mathrm{K}_*^G(X)$$

which are natural under equivariant continuous coarse maps and make the diagrams

$$\begin{array}{ccc} \mathrm{K}_*^G(X) & \xrightarrow{\mu_G} & \mathrm{K}_*(C_G^* X) \\ c_G \searrow & & \nearrow \mu_G \\ & \mathrm{KX}_*^G(X) & \end{array} \quad \begin{array}{ccc} \mathrm{K}_{1-*}(\mathfrak{c}^{\mathrm{red}} X \rtimes_\mu G) & \xrightarrow{\mu_G^*} & \mathrm{K}_*^G(X) \\ \mu_G^* \searrow & & \nearrow c_G^* \\ & \mathrm{KX}_*^G(X) & \end{array}$$

commute.

Inserting the equivariance into the constructions of 4.6.3 now gives rise to equivariant coarsified external and slant products

$$\begin{aligned} \times: & \quad \mathrm{KX}_m^G(X) \otimes \mathrm{KX}_n^H(Y) && \rightarrow \mathrm{KX}_{m+n}^{G \times H}(X \times Y) \\ \times: & \quad \mathrm{SX}_m^G(X) \otimes \mathrm{KX}_n^H(Y) && \rightarrow \mathrm{SX}_{m+n}^{G \times H}(X \times Y) \\ /: & \quad \mathrm{KX}_p^{G \times H}(X \times Y) \otimes \mathrm{KX}_q^H(Y) && \rightarrow \mathrm{KX}_{p-q}^G(X) \\ /: & \quad \mathrm{KX}_p^{G \times H}(X \times Y) \otimes \mathrm{K}_{1-q}(\mathfrak{c}^{\mathrm{red}} Y \rtimes_\mu H) && \rightarrow \mathrm{KX}_{p-q}^G(X) \\ /: & \quad \mathrm{SX}_p^{G \times H}(X \times Y) \otimes \mathrm{K}_{1-q}(\mathfrak{c}^{\mathrm{red}} Y \rtimes_\mu H) && \rightarrow \mathrm{SX}_{p-q}^G(X) \end{aligned}$$

and pairings

$$\begin{aligned} \langle -, - \rangle: \mathrm{KX}_p^H(Y) \otimes \mathrm{KX}_H^q(Y) &\rightarrow \mathrm{K}_{p-q}(\mathbb{C}) \\ \langle -, - \rangle: \mathrm{KX}_p^H(Y) \otimes \mathrm{K}_{1-q}(\mathfrak{c}^{\mathrm{red}}Y \rtimes_{\mu} H) &\rightarrow \mathrm{K}_{p-q}(\mathbb{C}) \end{aligned}$$

which are natural under pairs of equivariant coarse maps, compatible with the maps in the equivariant coarsified Higson–Roe sequence, the equivariant coarsification and co-coarsification maps and co-assembly.

Furthermore, if Z is a third proper metric space of bounded geometry with a proper isometric action by a countable discrete group K , then taking first the external product with an element $z \in \mathrm{KX}_m^{H \times K}(Y \times Z)$ and then the slant product with an element $\theta \in \mathrm{K}_{1-n}(\mathfrak{c}^{\mathrm{red}}Z \rtimes_{\mu} K)$ or $\theta \in \mathrm{KX}_K^n(Z)$ is equal to the external product with $z/\theta \in \mathrm{KX}_{m-n}^H(Y)$, and in particular if $Y = \{*\}$ is a one-point space trivially acted on by $H = 1$, then this composition is equal to multiplication with $\langle z, \theta \rangle = \langle z, \mu^*(\theta) \rangle = \langle \mu(z), \theta \rangle$, if Z has continuously bounded geometry.

5.5 Compatibility with the non-equivariant version

In this section, we describe two ways in which the equivariant slant products from Section 5 are compatible with the construction in Section 4.

To formulate the first type of compatibility concisely, we use the symbol $\mathrm{C}_?^*$ as a placeholder to denote either C^* , C_L^* or $\mathrm{C}_{L,0}^*$. We consider the canonical forgetful map

$$\mathrm{F}: \mathrm{C}_{G,?}^*X \rightarrow \mathrm{C}_?^*X$$

which is the inclusion map if $\mathrm{C}_{G,?}^*X$ is defined on a given (X, G) -module and $\mathrm{C}_?^*X$ is defined on the underlying X -module. On the level of K-theory, this induces a map

$$\mathrm{F}_*: \mathcal{HR}_*^G(X) \rightarrow \mathcal{HR}_*(X),$$

where we use the notation from Remark 1.12.

Recall that in Section 4 we have constructed slant products

$$/: \mathcal{HR}_p(X \times Y) \otimes \mathrm{K}_{1-q}(\mathfrak{c}^{\mathrm{red}}Y) \rightarrow \mathcal{HR}_{p-q}(X)$$

and from Section 5 we have

$$/: \mathcal{HR}_p^{G \times H}(X \times Y) \otimes \mathrm{K}_{1-q}(\mathfrak{c}^{\mathrm{red}}Y \rtimes_{\mu} H) \rightarrow \mathcal{HR}_{p-q}^G(X).$$

Moreover, there is the $*$ -homomorphism

$$\iota: \mathfrak{c}^{\mathrm{red}}Y \rightarrow \mathfrak{c}^{\mathrm{red}}Y \rtimes_{\mu} H, \quad f \mapsto f\delta_e,$$

which is well-defined because H is discrete. Then we have the following observation:

Proposition 5.30. *Let $x \in \mathcal{HR}_p^{G \times H}(X \times Y)$ and $y \in \mathrm{K}_{1-q}(\mathfrak{c}^{\mathrm{red}}Y)$. Then*

$$\mathrm{F}_*(x/\iota_*(y)) = \mathrm{F}_*(x)/y \in \mathcal{HR}_{p-q}(X).$$

Proof. Follows directly by comparing definitions. □

The second and more intricate type of compatibility deals with the case of a free action. Indeed, if G acts freely on X , then there are canonical induction isomorphisms for K-theory

$$\mathrm{Z}_X: \mathrm{K}_G^*(X) \xrightarrow{\cong} \mathrm{K}^*(G \backslash X), \tag{5.11}$$

and K-homology

$$l_X : K_*^G(X) \xrightarrow{\cong} K_*(G \backslash X). \quad (5.12)$$

The isomorphism (5.11) follows from a canonical Morita equivalence given by Green's imprimitivity theorem, see [Wil07, Corollary 4.11] for a textbook reference. For (5.12) see [WY19, Theorem 6.5.15]. The main result of this subsection is the following comparison theorem.

Theorem 5.31. *Let X and Y be proper metric spaces which are endowed with proper and free actions of countable discrete groups G and H , respectively. Then the equivariant slant product on K-homology agrees with the usual slant product on the corresponding quotient spaces up to the isomorphisms (5.11) and (5.12). In other words, the following diagram commutes.*

$$\begin{array}{ccc} K_p^{G \times H}(X \times Y) \otimes K_H^q(Y) & \xrightarrow{\quad / \quad} & K_{p-q}^G(X) \\ \downarrow l_{X \times Y} \otimes Z_Y & & \downarrow l_X \\ K_p(G \backslash X \times H \backslash Y) \otimes K^q(H \backslash Y) & \xrightarrow{\quad / \quad} & K_{p-q}(G \backslash X) \end{array}$$

In view of Lemma 4.19 and our definitions of equivariant K-homology and K-theory, Theorem 5.31 is equivalent to the following technical proposition.

Proposition 5.32. *Suppose we are in the setup of Theorem 5.31. Then the equivariant slant product from Definition 5.22 agrees with the description of the slant product in Lemma 4.19 up to the isomorphisms (5.11) and (5.12). In other words, the following diagram commutes.*

$$\begin{array}{ccc} K_p(C_{G \times H, L}^*(X \times Y)) \otimes K_{-q}(C_0(Y) \rtimes H) & \xrightarrow{5.22} & K_{p-q}(C_{G, L}^* X) \\ \downarrow l_{X \times Y} \otimes Z_Y & & \downarrow l_X \\ K_p(C_L^*(G \backslash X \times H \backslash Y)) \otimes K_{-q}(C_0(H \backslash Y)) & \xrightarrow{4.19} & K_{p-q}(C_L^*(G \backslash X)) \end{array}$$

To prove Proposition 5.32, we need an explicit description of the induction isomorphisms (5.11) and (5.12). Start with an explicit Morita equivalence which implements (5.11). We exhibit the construction from [Wil07, Corollary 4.11] for convenience of the reader. The expression

$$\langle f | f' \rangle_{C_0(G \backslash X)}(Gx) := \sum_{g \in G} \overline{f(g^{-1}y)} f'(g^{-1}y), \quad f, f' \in C_c(X),$$

yields a (right) inner product on $C_c(X)$ with values in $C_c(G \backslash X)$. Moreover, $C_c(G \backslash X)$ acts on $C_c(X)$ from the right by

$$(f \cdot g)(x) = f(x)g(Gx), \quad f \in C_c(X), g \in C_c(G \backslash X).$$

Similarly, there is a left inner product on $C_c(X)$ with values in $C_c(X) \rtimes_{\text{alg}} G$ defined by

$${}_{C_0(X) \rtimes G} \langle f' | f \rangle := \sum_{g \in G} f' \overline{g \cdot f} \delta_g, \quad f', f \in C_c(X).$$

and the crossed product $C_c(X) \rtimes_{\text{alg}} G$ acts from the left on $C_c(X)$ by

$$\left(\sum_{g \in G} f_g \delta_g \right) \cdot f = \sum_{g \in G} f_g (g \cdot f).$$

The right action of $C_c(G \setminus X)$ commutes with the left action of $C_c(X) \rtimes_{\text{alg}} G$. Simultaneously completing $C_c(X)$ and the coefficient algebras, we obtain an $C_0(X) \rtimes G$ - $C_0(G \setminus X)$ -Morita equivalence bimodule which we denote by Z_X . The isomorphism (5.11) is given by the $*$ -isomorphism

$$\phi: C_0(X) \rtimes G \xrightarrow{\cong} \mathcal{K}_{C_0(G \setminus X)}(Z_X)$$

that is induced by the left action.

We denote the conjugate module of Z_X by \bar{Z}_X . Then we have canonical identifications

$$Z_Y \otimes_{C_0(G \setminus X)} \bar{Z}_X \cong \mathcal{K}_{C_0(G \setminus X)}(Z_X) \cong C_0(X) \rtimes G, \quad (5.13)$$

see [BLM04, Corollary 8.2.15] for the first isomorphism. Similarly,

$$\bar{Z}_X \otimes_{C_0(X) \rtimes G} Z_X \cong \mathcal{K}_{C_0(X) \rtimes G}(\bar{Z}_X) \cong C_0(G \setminus X). \quad (5.14)$$

Next, we describe the isomorphism (5.12) in a way that is compatible with the above Morita equivalence. If we start with an ample $G \setminus X$ -module $H_{G \setminus X}$, then $Z_X \otimes_{C_0(G \setminus X)} H_{G \setminus X}$ is an ample (X, G) -module.³³ Similarly, if H_X is an ample (X, G) -module, then $\bar{Z}_X \otimes_{C_0(X) \rtimes G} H_X$ is an ample $G \setminus X$ -module. Moreover, by (5.13) and (5.14) these constructions are mutually inverse up to canonical isomorphisms. Hence we will assume that the representations on H_X and $H_{G \setminus X}$ have been chosen in such a way that we can identify $H_X = Z_X \otimes_{C_0(G \setminus X)} H_{G \setminus X}$ and thus $H_{G \setminus X} = \bar{Z}_X \otimes_{C_0(X) \rtimes G} H_X$. We introduce the following auxiliary concept to describe (5.12).

Definition 5.33. Let $(S_t) \in C_L^*(G \setminus X)$ and $(T_t) \in C_{G,L}^*(X)$. We say that (T_t) is a *lift* of (S_t) (or, alternatively, (S_t) is *pushdown* of (T_t)) if for each $z \in Z_X$ we have

$$\begin{aligned} (\mathfrak{T}_z \circ S_t - T_t \circ \mathfrak{T}_z)_{t \in [1, \infty)} &\in C_0([1, \infty), \mathcal{B}(H_{G \setminus X}, H_X)), \\ (S_t \circ \mathfrak{T}_z^* - \mathfrak{T}_z^* \circ T_t)_{t \in [1, \infty)} &\in C_0([1, \infty), \mathcal{B}(H_X, H_{G \setminus X})), \end{aligned}$$

where

$$\mathfrak{T}_z: H_{G \setminus X} \rightarrow H_X = Z_X \otimes_{C_0(G \setminus X)} H_{G \setminus X}, \quad \xi \mapsto z \otimes \xi. \quad \blacklozenge$$

Remark 5.34 (Notation). To work with the defining conditions of lifts and push-downs more conveniently, we will use the following notation: For two families of linear operators $(T_t)_{t \in [1, \infty)}$ and $(\tilde{T}_t)_{t \in [1, \infty)}$ with the same domain and target space, we write $T_t \sim \tilde{T}_t$ if $T_t - \tilde{T}_t \rightarrow 0$ in operator norm. Then the conditions from Definition 5.33 are equivalent to $\mathfrak{T}_z \circ S_t \sim T_t \circ \mathfrak{T}_z$ and $S_t \circ \mathfrak{T}_z^* \sim \mathfrak{T}_z^* \circ T_t$. \blacklozenge

In our constructions below, we will use the following ideal in the localization algebra.

$$N_{G,L}^*(X) := \{(T_t) \in C_{G,L}^*(X) \mid \forall f \in C_0(X): \lim_{t \rightarrow \infty} T_t \rho_X(f) = \lim_{t \rightarrow \infty} \rho_X(f) T_t = 0\}$$

An Eilenberg swindle as in [WY19, Lemma 6.4.11] shows that the K-theory of $N_{G,L}^*(X)$ vanishes. Hence the canonical projection $C_{G,L}^*(X) \rightarrow C_{G,L}^*(X)/N_{G,L}^*(X)$ induces an isomorphism on K-theory.

Lemma 5.35.

- (i) *Every $(S_t) \in C_L^*(G \setminus X)$ admits a lift and every $(T_t) \in C_{G,L}^*(X)$ admits a pushdown.*

³³The proof that H_X constructed in this way is ample can be reduced to the trivial product situation $X = G \times \underline{X}$, compare the proof of Lemma 5.35.

(ii) Let $(T_t) \in C_{G,L}^*(X)$ be a lift of $(S_t) \in C_L^*(G \setminus X)$. Then we have $(T_t) \in N_{G,L}^*(X)$ if and only if $(S_t) \in N_L^*(G \setminus X)$.

Proof. (i) The argument here is essentially the same construction as in [WY19, Construction 6.5.14]. Let $U \subseteq G \setminus X$ be an open subset over which the canonical projection $\pi: X \rightarrow G \setminus X$ is trivialized, that is, $\pi^{-1}(U) \cong G \times U$. Let $Z_{\pi^{-1}(U)}$ be the closure of $C_0(\pi^{-1}(U)) \cdot Z_X$. Note that this is at the same time the closure of $Z_X \cdot C_0(U)$ and this module implements the Morita equivalence from $C_0(\pi^{-1}(U)) \rtimes G$ to $C_0(U)$. Since $\pi^{-1}(U) \cong G \times U$, we have

$$Z_{\pi^{-1}(U)} \cong \ell^2(G) \otimes C_0(U) \quad (5.15)$$

and

$$1_{\pi^{-1}(U)} H_X = Z_{\pi^{-1}(U)} \otimes_{C_0(U)} 1_U H_{G \setminus X} \cong \ell^2(G) \otimes 1_U H_{G \setminus X} \quad (5.16)$$

Suppose for the moment that S_t has support inside U . That is, there exists a continuous function $\chi: G \setminus X \rightarrow [0, 1]$ with $\text{supp}(\chi) \subseteq U$ such that $\chi S_t = S_t \chi = S_t$. Then set $T_t := 1_{\pi^{-1}(U)} (\text{id}_{\ell^2(G)} \otimes S_t) 1_{\pi^{-1}(U)} \in C_{G,L}^*(X)$, where we implicitly use (5.16). We claim that T_t is a lift of S_t . Let $z \in C_c(X) \subseteq Z_X$. Then, by construction, $T_t \circ \mathfrak{T}_z = T_t \circ \mathfrak{T}_{(\chi \circ \pi)z}$, $\mathfrak{T}_z^* \circ T_t = \mathfrak{T}_{(\chi \circ \pi)z}^* \circ T_t$, $\mathfrak{T}_z \circ S_t = \mathfrak{T}_{(\chi \circ \pi)z} \circ S_t$ and $S_t \circ \mathfrak{T}_z^* = S_t \circ \mathfrak{T}_{(\chi \circ \pi)z}^*$. Hence it suffices to check the defining condition of a lift for $z \in C_c(\pi^{-1}(U)) \subseteq Z_{\pi^{-1}(U)} \subseteq Z_X$. In view of (5.15), we are then further reduced to checking it for elements of the form $z = \delta_g \otimes \varphi$, where $g \in G$, $\varphi \in C_c(U)$. Then

$$\mathfrak{T}_z \circ S_t - T_t \circ \mathfrak{T}_z = (\xi \mapsto \delta_g \otimes \underbrace{[\varphi, S_t]}_{\rightarrow 0} \xi) \rightarrow 0$$

considered as a map $H_{G \setminus X} \rightarrow \ell^2(G) \otimes 1_U H_{G \setminus X} \cong 1_{\pi^{-1}(U)} H_X \subseteq H_X$. Similarly,

$$S_t \circ \mathfrak{T}_z^* - \mathfrak{T}_z^* \circ T_t = (\delta_\gamma \otimes \xi \mapsto \langle \delta_g | \delta_\gamma \rangle \underbrace{[S_t, \varphi]}_{\rightarrow 0} \xi) \circ 1_{\pi^{-1}(U)} \rightarrow 0,$$

where the right-hand side is viewed as a composition

$$H_X \xrightarrow{1_{\pi^{-1}(U)}} 1_{\pi^{-1}(U)} H_X \cong \ell^2(G) \otimes 1_U H_{G \setminus X} \rightarrow H_{G \setminus X}.$$

This shows that (T_t) is a lift of (S_t) .

In general, we take a locally finite covering \mathcal{U} of $G \setminus X$ such that $\pi: X \rightarrow G \setminus X$ is trivial over each $U \in \mathcal{U}$. Then choose an ℓ^2 -partition³⁴ of unity $(\phi_U)_{U \in \mathcal{U}}$ subordinate to \mathcal{U} . Then for each $U \in \mathcal{U}$, the element $(\phi_U S_t \phi_U)$ is supported inside U . Take the corresponding lift which was constructed in the previous paragraph and denote it by $(T_t^U) \in C_{G,L}^*(X)$. Then define $T_t := \sum_{U \in \mathcal{U}} T_t^U$. This is a strongly converging sum since the covering \mathcal{U} is locally finite and one can easily verify that $(T_t) \in C_{G,L}^*(X)$. We again claim that (T_t) is the desired lift of (S_t) . To prove this, now take $z \in Z_X$, $\varphi \in C_c(G \setminus X)$. Then

$$\mathfrak{T}_{z\varphi} \sum_{U \in \mathcal{U}} \phi_U S_t \phi_U = \mathfrak{T}_z \sum_{U \in \mathcal{U}} \varphi \phi_U S_t \phi_U \sim \mathfrak{T}_z \sum_{U \in \mathcal{U}} \varphi \phi_U^2 S_t = \mathfrak{T}_z \varphi S_t = \mathfrak{T}_{z\varphi} S_t,$$

where we use that $[S_t, \phi_U] \rightarrow 0$ and the fact that the sums in the middle terms have only finitely many non-zero entries because the covering \mathcal{U} is locally finite. Thus

$$\mathfrak{T}_{z\varphi} \circ S_t \sim \sum_{U \in \mathcal{U}} \mathfrak{T}_{z\varphi} \circ \phi_U S_t \phi_U \sim \sum_{U \in \mathcal{U}} T_t^U \circ \mathfrak{T}_{z\varphi} = T_t \circ \mathfrak{T}_{z\varphi},$$

³⁴That means that (ϕ_U^2) is a partition of unity.

where we again have used that the sums have only finitely many non-zero entries. Since elements of the form $z\varphi$ are dense in Z_X , we actually obtain

$$\mathfrak{T}_z \circ S_t \sim T_t \circ \mathfrak{T}_z$$

for all $z \in Z_X$. An analogous argument shows that $S_t \circ \mathfrak{T}_z^* \sim \mathfrak{T}_z^* \circ T_t$. This completes the proof that (T_t) is a lift of (S_t) .

The existence of pushdowns is proved analogously by first solving the problem for operators which are supported inside $\pi^{-1}(U) \cong G \times U$. Indeed, if an operator on $\ell^2(G) \otimes 1_U H_{G \setminus X} \subseteq H_X$ is G -equivariant and has sufficiently small propagation in X , then it is necessarily of the form $\text{id} \otimes S$, and hence has a pushdown. The general case can again be reduced to this via an ℓ^2 -partition of unity.

(ii) Suppose that $(S_t) \in \mathcal{N}_L^*(G \setminus X)$ and $(T_t) \in \mathcal{C}_{G,L}^*(X)$ is a lift of (S_t) . Let $z_1, z_2 \in Z_X$ and $\varphi \in C_0(G \setminus X)$. Then with $f := {}_{C_0(X) \rtimes G} \langle z_1 \varphi \mid z_2 \rangle \in C_0(X) \rtimes G$, we have

$$fT_t = \mathfrak{T}_{z_1\varphi} \circ \mathfrak{T}_{z_2}^* \circ T_t \sim \mathfrak{T}_{z_1\varphi} \circ S_t \circ \mathfrak{T}_{z_2}^* = \mathfrak{T}_{z_1} \circ \underbrace{\varphi S_t}_{\rightarrow 0} \circ \mathfrak{T}_{z_2}^* \rightarrow 0.$$

and

$$T_t f = T_t \circ \mathfrak{T}_{z_1\varphi} \circ \mathfrak{T}_{z_2}^* \sim \mathfrak{T}_{z_1\varphi} \circ S_t \circ \mathfrak{T}_{z_2}^* = \mathfrak{T}_{z_1} \circ \underbrace{\varphi S_t}_{\rightarrow 0} \circ \mathfrak{T}_{z_2}^* \rightarrow 0.$$

Since the left-module structure on Z_Y is full, this is enough to check that $(T_t) \in \mathcal{N}_{G,L}^*(X)$. The converse implication is proved analogously. \square

The following proposition is an immediate consequence of Lemma 5.35.

Proposition 5.36. *Choosing pushdowns induces a well-defined $*$ -isomorphism*

$$\mathfrak{l}_X: \frac{\mathcal{C}_{G,L}^*(X)}{\mathcal{N}_{G,L}^*(X)} \xrightarrow{\cong} \frac{\mathcal{C}_L^*(G \setminus X)}{\mathcal{N}_L^*(G \setminus X)},$$

with inverse given by choosing lifts. Together with the canonical isomorphisms $\mathcal{K}_*(\mathcal{C}_{-,L}^*(-) \rightarrow \mathcal{K}_*(\mathcal{C}_{-,L}^*(-)/\mathcal{N}_{-,L}^*(-)))$, this yields the isomorphism (5.12).

Since the construction of lifts and pushdowns in Lemma 5.35 is the same as in [WY19, Construction 6.5.14], the isomorphism \mathfrak{l}_X is indeed the same as the one given by [WY19, Theorem 6.5.15].

To compose our slant product with the Morita equivalence, we need a slightly more general version of the construction from Definition 5.22. To that end, let us fix a—for the moment arbitrary—countably generated Hilbert $C_0(Y) \rtimes H$ -module E . We will construct a $*$ -homomorphism

$$\Upsilon_L: \mathcal{C}_{G \times H, L}^*(X \times Y) \otimes_{\max} \mathcal{K}_{C_0(Y) \rtimes H}(E) \rightarrow \frac{\mathcal{C}_{G,L}^*(\tilde{\rho}_X)}{C_0([1, \infty), \mathcal{C}_G^*(\tilde{\rho}_X))}, \quad (5.17)$$

where we redefine $\tilde{H}_X := H_X \otimes (E \otimes_{C_0(Y) \rtimes H} H_Y)$ and

$$\tilde{\rho}_X \rtimes \tilde{u}_G := (\rho_X \rtimes u_G) \otimes \text{id} \otimes \text{id}: C_0(X) \rtimes G \rightarrow \mathcal{B}(\tilde{H}_X).$$

Note that if $E = (C_0(Y) \rtimes H) \otimes \ell^2$ and we also replace H_Y by $\ell^2(H) \otimes H_Y$, then this reduces to our previous definitions. There is a canonical $*$ -homomorphism

$$\kappa_E: \mathcal{K}_{C_0(Y) \rtimes H}(E) \rightarrow \mathcal{E}_{G,L}^*(\tilde{\rho}_X), \quad K \mapsto (\text{id}_{H_X} \otimes K \otimes_{C_0(Y) \rtimes H} \text{id}_{H_Y}),$$

viewed as constant functions. Unfortunately, in general there is no canonical map $\mathcal{C}_{G \times H, L}^*(X \times Y) \rightarrow \mathcal{E}_{G,L}^*(\tilde{\rho}_X)$ which would be required to precisely mimic Definition 5.22. Instead, we use a similar scheme as in the construction of the induction

isomorphisms. The next definition, along with its name, is inspired by the notion of “connection” which appears in the construction of the Kasparov product (see for instance [Bla98, Section 18.3]).

Definition 5.37. Let $(T_t) \in C_{G \times H, L}^*(X \times Y)$. We say $(F_t) \in E_{G, L}^*(\tilde{\rho}_X)$ is a *connection* for (T_t) , if for each $K \in \mathcal{K}_{C_0(Y) \rtimes H}(E)$, we have

$$\begin{aligned} (F_t \circ \kappa_E(K))_{t \in [1, \infty)} &\in C_{G, L}^*(\tilde{\rho}_X), \\ (\kappa_E(K) \circ F_t)_{t \in [1, \infty)} &\in C_{G, L}^*(\tilde{\rho}_X), \end{aligned}$$

and for each $e \in E$ we have

$$\begin{aligned} (\mathfrak{T}_e \circ T_t - F_t \circ \mathfrak{T}_e)_{t \in [1, \infty)} &\in C_0([1, \infty), \mathcal{B}(H_X \otimes H_Y, \tilde{H}_X)), \\ (T_t \circ \mathfrak{T}_e^* - \mathfrak{T}_e^* \circ F_t)_{t \in [1, \infty)} &\in C_0([1, \infty), \mathcal{B}(\tilde{H}_X, H_X \otimes H_Y)), \end{aligned}$$

where

$$\mathfrak{T}_e: H_X \otimes H_Y \rightarrow \tilde{H}_X \cong E \otimes_{\text{id} \otimes (C_0(Y) \rtimes H)} (H_X \otimes H_Y), \quad \xi \mapsto e \otimes \xi. \quad \blacklozenge$$

Lemma 5.38.

- (i) Every $(T_t) \in C_{G \times H, L}^*(X \times Y)$ admits a connection $(F_t) \in E_{G, L}^*(\tilde{\rho}_X)$.
- (ii) If $(F_t) \in E_{G, L}^*(\tilde{\rho}_X)$ is a connection for $(T_t) \in C_{G \times H, L}^*(X \times Y)$ and $K \in \mathcal{K}_{C_0(Y) \rtimes H}(E)$, then $[F_t, \kappa_E(K)] \in C_0([1, \infty), C_G^*(\tilde{\rho}_X))$.
- (iii) Let $(F_t) \in E_{G, L}^*(\tilde{\rho}_X)$ be a connection for $(T_t) \in C_{G \times H, L}^*(X \times Y)$ and $K \in \mathcal{K}_{C_0(Y) \rtimes H}(E)$.
 - If $(T_t) \in C_0([1, \infty), C_{G \times H}^*(X \times Y))$, then

$$(F_t \circ \kappa_E(K)) \in C_0([1, \infty), C_G^*(\tilde{\rho}_X)).$$

- If $(T_t) \in N_{G \times H, L}^*(X \times Y)$, then $(F_t \circ \kappa_E(K)) \in N_{G, L}^*(\tilde{\rho}_X)$.

Proof. (i) We start with the case that $E = (C_0(Y) \rtimes H) \otimes \ell^2$ is the standard Hilbert module over $C_0(Y) \rtimes H$. In this case, $\tilde{H}_X = H_X \otimes H_Y \otimes \ell^2$ and a connection for $(T_t) \in C_{G \times H, L}^*(X \times Y)$ is given by $F_t := T_t \circ \text{id}_{\ell^2}$. This is because for $e = f \otimes v \in E = (C_0(Y) \rtimes H) \otimes \ell^2$, we have

$$\mathfrak{T}_e \circ T_t - F_t \circ \mathfrak{T}_e = (\xi \mapsto \underbrace{[(\text{id} \otimes f), T_t]}_{\rightarrow 0} \xi \otimes v) \rightarrow 0$$

as an operator $H_X \otimes H_Y \rightarrow H_X \otimes H_Y \otimes \ell^2$ and, similarly, $T_t \circ \mathfrak{T}_e^* - \mathfrak{T}_e^* \circ F_t \rightarrow 0$. Moreover, if $K = f \otimes L \in \mathcal{K}_{C_0(Y) \rtimes H}(E) \cong (C_0(Y) \rtimes H) \otimes \mathcal{K}(\ell^2)$, then

$$F_t \circ \kappa_E(K) = (T_t \circ (\text{id}_{H_X} \otimes f)) \otimes L \in C_{G, L}^*(\tilde{\rho}_X)$$

and similarly for $\kappa_E(K) \circ F_t$.

In general, we apply Kasparov’s stabilization theorem to embed E into $(C_0(Y) \rtimes H) \otimes \ell^2$ such that there exists an adjointable projection P on $(C_0(Y) \rtimes H) \otimes \ell^2$ whose image is E . To complete the argument, observe that if (\hat{F}_t) is a connection for (T_t) with respect to the standard Hilbert module, then $F_t := P \hat{F}_t P$ yields a connection with respect to E .

(ii) By the definition of the compact operators on a Hilbert module, the image of κ_E is the closed linear span of elements $\mathfrak{T}_{e_1} \circ \mathfrak{T}_{e_2}^* = \kappa_E(|e_1\rangle\langle e_2|)$, where $e_1, e_2 \in E$. Hence the desired statement follows from the defining conditions of a connection:

$$F_t \circ \mathfrak{T}_{e_1} \circ \mathfrak{T}_{e_2}^* \sim \mathfrak{T}_{e_1} \circ T_t \circ \mathfrak{T}_{e_2}^* \sim \mathfrak{T}_{e_1} \circ \mathfrak{T}_{e_2}^* \circ F_t.$$

(iii) Suppose that $(T_t) \in C_0([1, \infty), C_{G \times H}^*(X \times Y))$. As in the previous part, we can assume that $\kappa_E(K) = \mathfrak{T}_{e_1} \circ \mathfrak{T}_{e_2}^*$. Then

$$F_t \circ \kappa_E(K) = F_t \circ \mathfrak{T}_{e_1} \circ \mathfrak{T}_{e_2}^* \sim \mathfrak{T}_{e_1} \circ \underbrace{T_t}_{\rightarrow 0} \circ \mathfrak{T}_{e_2}^* \rightarrow 0,$$

as required. If $(T_t) \in N_{G \times H, L}^*(X \times Y)$, then let $\varphi \in C_0(Y)$ and assume $\kappa_E(K) = \mathfrak{T}_{e_1 \varphi} \circ \mathfrak{T}_{e_2}^*$. This is justified because $C_0(Y)$ contains an approximate identity of $C_0(Y) \rtimes H$. Then for each $f \in C_0(X)$, we have

$$\begin{aligned} \tilde{\rho}_X(f) \circ F_t \circ \kappa_E(K) &= \tilde{\rho}_X(f) \circ F_t \circ \mathfrak{T}_{e_1 \varphi} \circ \mathfrak{T}_{e_2}^* \\ &\sim \tilde{\rho}_X(f) \circ \mathfrak{T}_{e_1 \varphi} \circ T_t \circ \mathfrak{T}_{e_2}^* \\ &= \mathfrak{T}_{e_1} \circ \underbrace{(f \otimes \varphi) T_t}_{\rightarrow 0} \circ \mathfrak{T}_{e_2}^* \rightarrow 0 \end{aligned}$$

and hence $F_t \circ \kappa_E(K) \in N_{G, L}^*(\tilde{\rho}_X)$. \square

We now can define (5.17) by

$$(T_t) \otimes K \mapsto [(F_t \circ \kappa_E(K))_{t \in [1, \infty)}],$$

where we choose a connection (F_t) for each (T_t) . This is well-defined by Lemma 5.38. In fact, it shows that this yields a well-defined $*$ -homomorphism

$$\Upsilon_L: \frac{C_{G \times H, L}^*(X \times Y)}{N_{G \times H, L}^*(X \times Y)} \otimes_{\max} \mathcal{K}_{C_0(Y) \rtimes H}(E) \rightarrow \frac{C_{G, L}^*(\tilde{\rho}_X)}{N_{G, L}^*(\tilde{\rho}_X)}. \quad (5.18)$$

Since the K-theory of N_L^* vanishes, this still defines a map

$$K_p(C_{G \times H, L}^*(X \times Y)) \otimes K_{-q}(\mathcal{K}_{C_0(Y) \rtimes H}(E)) \rightarrow K_{p-q}(C_{G, L}^*(X)) \quad (5.19)$$

by a completely analogous recipe as Definition 5.22. If E is a full Hilbert $C_0(Y) \rtimes H$ -module, then $K_*(\mathcal{K}_{C_0(Y) \rtimes H}) \cong K_*(C_0(Y) \rtimes H)$. In this case, it can be verified that (5.19) agrees with the previous definition 5.22 by embedding E into the standard module $(C_0(Y) \rtimes H) \otimes \ell^2$ via the Kasparov stabilization theorem and replacing H_Y by $\ell^2(H) \otimes H_Y$.

Proof of Proposition 5.32. We start with an arbitrary countably generated Hilbert- $C_0(Y) \rtimes H$ -module E_Y that is full. Then let $E_{H \setminus Y} := E_Y \otimes_{C_0(Y) \rtimes H} Z_Y$ and $\phi: \mathcal{K}_{C_0(Y) \rtimes H}(E_Y) \rightarrow \mathcal{K}_{C_0(H \setminus Y)}(E_{H \setminus Y})$ be the canonical map $S \mapsto S \otimes \text{id}$. The map ϕ implements the isomorphism (5.11) in this setup. In view of the previous discussion, it suffices to prove that the following diagram commutes.

$$\begin{array}{ccc} \frac{C_{G \times H, L}^*(X \times Y)}{N_{G \times H, L}^*(X \times Y)} \otimes_{\max} \mathcal{K}_{C_0(Y) \rtimes H}(E_Y) & \xrightarrow{\Upsilon_L} & \frac{C_{G, L}^*(\tilde{\rho}_X)}{N_{G, L}^*(\tilde{\rho}_X)} \\ \downarrow \downarrow_{X \times Y} \otimes \phi & & \downarrow \downarrow_X \\ \frac{C_L^*(G \setminus X \times H \setminus Y)}{N_L^*(G \setminus X \times H \setminus Y)} \otimes_{\max} \mathcal{K}_{C_0(H \setminus Y)}(E_{H \setminus Y}) & \xrightarrow{\Upsilon_L} & \frac{C_L^*(\tilde{\rho}_{G \setminus X})}{N_L^*(\tilde{\rho}_{G \setminus X})} \end{array}$$

Note that we have a canonical identification

$$\begin{aligned}
E_{H \setminus Y} \otimes_{C_0(H \setminus Y)} H_{H \setminus Y} &= (E_Y \otimes_{C_0(Y) \rtimes H} Z_Y) \otimes_{C_0(H \setminus Y)} H_{H \setminus Y} \\
&= E_Y \otimes_{C_0(Y) \rtimes H} (Z_Y \otimes_{C_0(H \setminus Y)} H_{H \setminus Y}) \\
&= E_Y \otimes_{C_0(Y) \rtimes H} H_Y
\end{aligned} \tag{5.20}$$

and hence

$$\begin{aligned}
Z_X \otimes_{C_0(G \setminus X)} \tilde{H}_{G \setminus X} &= (Z_X \otimes_{C_0(G \setminus X)} H_{G \setminus X}) \otimes (E_{H \setminus Y} \otimes_{C_0(H \setminus Y)} H_{H \setminus Y}) \\
&= H_X \otimes (E_Y \otimes_{C_0(Y) \rtimes H} H_Y) = \tilde{H}_X.
\end{aligned}$$

Therefore it makes sense to consider the arrows entering the lower right corner to land in the same C*-algebra.

Now let $(T_t) \in C_{G \times H, L}^*(X \times Y)$ and $K \in \mathcal{K}_{C_0(Y) \rtimes H}(E_Y)$. Choose a pushdown $(S_t) \in C_L^*(G \setminus X \times H \setminus Y)$ for (T_t) . Moreover, choose a connection $(F_t) \in E_{G, L}^*(\tilde{\rho}_X)$ for (T_t) and a connection $(G_t) \in E_L^*(\tilde{\rho}_{G \setminus X})$ for (S_t) . To prove the desired commutativity, we need to verify that $G_t \circ \kappa_{E_{H \setminus Y}}(\phi(K))$ is a pushdown of $F_t \circ \kappa_{E_Y}(K)$. First observe that (5.20) identifies $\phi(K) \otimes_{C_0(H \setminus Y)} \text{id}$ with $K \otimes_{C_0(Y) \rtimes H} \text{id}$. Then let $z \in Z_X$, $w \in Z_Y$, $e \in E_Y$ and fix the following notation as in Definitions 5.33 and 5.37.

$$\begin{aligned}
\mathfrak{T}_z &: H_{G \setminus X} \rightarrow H_X \\
\mathfrak{T}_w &: H_{H \setminus Y} \rightarrow H_Y \\
\mathfrak{T}_e &: H_Y \rightarrow E_Y \otimes_{C_0(Y) \rtimes H} H_Y
\end{aligned}$$

We have $e \otimes w \in E_Y \otimes_{C_0(Y) \rtimes H} Z_Y = E_{H \setminus Y}$ and up to the identification (5.20) the following maps are equal.

$$\mathfrak{T}_{e \otimes w} = \mathfrak{T}_e \circ \mathfrak{T}_w: H_{H \setminus Y} \rightarrow E_{H \setminus Y} \otimes_{C_0(H \setminus Y)} H_{H \setminus Y} = E_Y \otimes_{C_0(Y) \rtimes H} H_Y$$

As before, the image of κ_{E_Y} is the closed linear span of operators of the form $\mathfrak{T}_{e_1} \circ \mathfrak{T}_{e_2}^*$, where $e_1, e_2 \in E_Y$. Thus we can assume without loss of generality that

$$K \otimes_{C_0(Y) \rtimes H} \text{id}_{H_Y} = \phi(K) \otimes_{C_0(H \setminus Y)} \text{id}_{H_Y} = \mathfrak{T}_{e_1} \circ \mathfrak{T}_{e_2}^*.$$

Hence

$$\begin{aligned}
F_t \circ \kappa_{E_Y}(K) \circ (\mathfrak{T}_z \otimes \text{id}) &= F_t \circ (\text{id} \otimes \mathfrak{T}_{e_1} \circ \mathfrak{T}_{e_2}^*) \circ (\mathfrak{T}_z \otimes \text{id}) \\
&= F_t \circ (\text{id} \otimes \mathfrak{T}_{e_1}) \circ (\mathfrak{T}_z \otimes \mathfrak{T}_{e_2}^*) \\
&\sim (\text{id} \otimes \mathfrak{T}_{e_1}) \circ T_t \circ (\mathfrak{T}_z \otimes \mathfrak{T}_{e_2}^*)
\end{aligned}$$

because (F_t) is a connection for (T_t) . We can furthermore assume that $e_2 = e'_2 \otimes_{C_0(Y) \rtimes H} \langle w' | w \rangle$, where $e'_2 \in E_Y$, $w, w' \in Z_Y$ because such elements are dense. Then $\mathfrak{T}_{e_2} = \mathfrak{T}_{e'_2} \circ \mathfrak{T}_{w'} \circ \mathfrak{T}_w^*$ and we continue

$$\begin{aligned}
&(\text{id} \otimes \mathfrak{T}_{e_1}) \circ T_t \circ (\mathfrak{T}_z \otimes \mathfrak{T}_{e_2}^*) \\
&= (\text{id} \otimes \mathfrak{T}_{e_1}) \circ T_t \circ (\mathfrak{T}_z \otimes \mathfrak{T}_w) \circ (\text{id} \otimes \mathfrak{T}_{w'}^* \circ \mathfrak{T}_{e'_2}^*) \\
&\sim (\text{id} \otimes \mathfrak{T}_{e_1}) \circ (\mathfrak{T}_z \otimes \mathfrak{T}_w) \circ S_t \circ (\text{id} \otimes \mathfrak{T}_{w'}^* \circ \mathfrak{T}_{e'_2}^*) \\
&= (\mathfrak{T}_z \otimes \text{id}) \circ (\text{id} \otimes \underbrace{\mathfrak{T}_{e_1} \circ \mathfrak{T}_w}_{\mathfrak{T}_{e_1 \otimes w}}) \circ S_t \circ (\text{id} \otimes \mathfrak{T}_{w'}^* \circ \mathfrak{T}_{e'_2}^*) \\
&\sim (\mathfrak{T}_z \otimes \text{id}) \circ G_t \circ (\text{id} \otimes \mathfrak{T}_{e_1} \circ \mathfrak{T}_w) \circ (\text{id} \otimes \mathfrak{T}_{w'}^* \circ \mathfrak{T}_{e'_2}^*) \\
&= (\mathfrak{T}_z \otimes \text{id}) \circ G_t \circ (\text{id} \otimes \mathfrak{T}_{e_1} \circ \mathfrak{T}_{e_2}^*) \\
&= (\mathfrak{T}_z \otimes \text{id}) \circ G_t \circ (\text{id} \otimes \phi(K) \otimes \text{id}) \\
&= (\mathfrak{T}_z \otimes \text{id}) \circ G_t \circ \kappa_{E_{H \setminus Y}}(\phi(K)).
\end{aligned}$$

because (S_t) is a pushdown of (T_t) and (G_t) is a connection for (S_t) . Hence

$$F_t \circ \kappa_{E_Y}(K) \circ (\mathfrak{T}_z \otimes \text{id}) \sim (\mathfrak{T}_z \otimes \text{id}) \circ G_t \circ \kappa_{E_{H \setminus Y}}(\phi(K)).$$

A completely analogous argument also proves that

$$(\mathfrak{T}_z^* \otimes \text{id}) \circ F_t \circ \kappa_{E_Y}(K) \sim G_t \circ \kappa_{E_{H \setminus Y}}(\phi(K)) \circ (\mathfrak{T}_z^* \otimes \text{id}).$$

Thus (G_t) is a pushdown of (F_t) , as required. \square

This finishes the proof of Theorem 5.31.

6 Injectivity of external products

In this section, we deduce our injectivity results for external product maps using the machinery developed in Sections 4 and 5. In the following statements, we fix an exact crossed product functor μ or $\mu = \text{red}$ if H is exact. Here and in the following, we will use the notation convention from Remark 1.12. We start with an injectivity result for the external product with a single element. This is a direct consequence of our equivariant slant products.

Theorem 6.1. *Let Y be a proper metric space of continuously bounded geometry endowed with a proper action of a countable discrete group H . Let $y \in K_n^H(Y)$ be such that there exists $\vartheta \in K_{1-n}(\mathbf{c}^{\text{red}}Y \rtimes_{\mu} H)$ with $\langle y, \mu_H^* \vartheta \rangle = 1$ (or with $\langle y, \mu_H^* \vartheta \rangle \neq 0$).*

Then for every proper metric space X endowed with a proper action of a countable discrete group G , the external product

$$\mathcal{HR}_*^G(X) \xrightarrow{\times y} \mathcal{HR}_{*+n}^{G \times H}(X \times Y)$$

is (rationally) split-injective.

Proof. It follows from Corollary 5.26 that a retraction for $\times y$ is given by (a rational multiple of) the map $\mathcal{HR}_{p+n}^{G \times H}(X \times Y) \xrightarrow{/\vartheta} \mathcal{HR}_p^G(X)$. \square

If the equivariant coarse co-assembly map is rationally surjective, then we obtain rational injectivity of the entire external product map in the presence of a free action on the second factor.

Theorem 6.2. *Let Y be a proper metric space of continuously bounded geometry endowed with a proper and free action of a countable discrete group H . Suppose that the equivariant coarse co-assembly map*

$$\mu_H^*: K_{1-*}(\mathbf{c}^{\text{red}}Y \rtimes_{\mu} H) \rightarrow K_H^*(Y)$$

is rationally surjective.

Then for every proper metric space X endowed with a proper action of a countable discrete group G , the external product

$$\mathcal{HR}_m^G(X) \otimes K_n^H(Y) \rightarrow \mathcal{HR}_{m+n}^{G \times H}(X \times Y)$$

is rationally injective for every $m, n \in \mathbb{Z}$.

Proof. First observe that the universal coefficient theorem implies that the pairing between K-theory and homology induces an isomorphism

$$K_n(H \setminus Y) \otimes \mathbb{Q} \cong \text{Hom}(K^n(H \setminus Y), \mathbb{Q}).$$

Since the action of H on Y is free, it follows from Definition 5.25 and Theorem 5.31 that the pairing between $K_n^H(Y)$ and $K_H^n(Y)$ is equivalent to the pairing between $K_n(H \setminus Y)$ and $K^n(H \setminus Y)$. Thus we also have an isomorphism

$$K_n^H(Y) \otimes \mathbb{Q} \cong \text{Hom}(K_H^n(Y), \mathbb{Q}). \quad (6.1)$$

Now pick a \mathbb{Q} -basis $(\theta_i)_{i \in I}$ for $K_H^n(Y) \otimes \mathbb{Q}$. By exploiting the rational surjectivity of the equivariant coarse co-assembly map and possibly multiplying the basis elements with integers, we can assume that there are elements $\vartheta_i \in K_{1-n}(\mathfrak{c}^{\text{red}}Y \rtimes_{\mu} H)$ with $\mu^*(\vartheta_i) = \theta_i$ for all $i \in I$. We use the equivariant slant products with all ϑ_i simultaneously. By Corollary 5.26, this yields a commutative diagram

$$\begin{array}{ccc} \mathcal{HR}_m^G(X) \otimes K_n^H(Y) \otimes \mathbb{Q} & \xrightarrow{\times} & \mathcal{HR}_{m+n}^{G \times H}(X \times Y) \otimes \mathbb{Q} \\ \cong \downarrow \text{id} \otimes \prod_{i \in I} \langle -, \theta_i \rangle & & \downarrow \prod_{i \in I} / \vartheta_i \\ \mathcal{HR}_m^G(X) \otimes \prod_{i \in I} \mathbb{Q} & \hookrightarrow & \prod_{i \in I} (\mathcal{HR}_m^G(X) \otimes \mathbb{Q}). \end{array}$$

The left vertical arrow is an isomorphism by (6.1) and the lower horizontal arrow is injective for abstract reasons. Consequently, this proves that the external product map is injective. \square

Remark 6.3. At first glance, one might also hope for a coarse version of the previous theorem using the definitions given in Section 4.6. That is, for a proper metric space Y of bounded geometry such that the coarse co-assembly map $\mu^*: K_{1-*}(\mathfrak{c}^{\text{red}}Y) \rightarrow KX^*(Y)$ is rationally surjective, the external product maps

$$\begin{aligned} \times: KX_m(X) \otimes KX_n(Y) &\rightarrow KX_{m+n}(X \times Y) \\ \times: SX_m(X) \otimes KX_n(X) &\rightarrow SX_{m+n}(X \times Y) \\ \times: K_m(C^*X) \otimes KX_n(Y) &\rightarrow K_{m+n}(C^*(X \times Y)) \end{aligned}$$

should be rationally injective.³⁵ However, there seems to be little hope. The proof of Theorem 6.2 does not work in this situation because the pairing between $KX_n(Y)$ and $KX^n(Y)$ can be degenerate in general. \blacklozenge

6.1 Proper metric spaces that are scaleable, combable or coarsely embeddable

In this section we show that in many situations, where one can prove the coarse Novikov conjecture, one can also prove injectivity of external products on the non-equivariant version of the Higson–Roe sequence. In the following, the space Y will come without a group action. In effect, we will apply Theorem 6.2 for the case that H is the trivial group.

We refrain from recalling here the notions occurring in the following corollary (like *scaleable* or *combing*) because they do not appear anywhere else in this paper. The interested reader can find the relevant definitions in the references provided in the proof below. The corollary is stated as Corollary 1.14 in the introduction.

³⁵The external product $\times: K_m(C^*X) \otimes KX_n(Y) \rightarrow K_{m+n}(C^*(X \times Y))$ is the external product of the Roe algebras composed with the coarsified assembly map $KX_n(Y) \rightarrow K_n(C^*Y)$.

Corollary 6.4. *Let Y be either*

- (a) *a uniformly contractible, proper metric space of continuously bounded geometry which is scaleable,*
- (b) *a uniformly contractible, proper metric space of continuously bounded geometry which admits an expanding and coherent combing, or*
- (c) *the universal cover EH of the classifying space BH of a group H , if BH is a finite complex and H is coarsely embeddable into a Hilbert space.*

Then for every proper metric space X endowed with a proper action of a countable discrete group G , the external product

$$\mathcal{HR}_m^G(X) \otimes K_n(Y) \rightarrow \mathcal{HR}_{m+n}^G(X \times Y)$$

is rationally injective for each $m, n \in \mathbb{Z}$.

Proof. In either case we will reduce to Theorem 6.2, where we take H to be the trivial group. Then we need to show that the ordinary coarse co-assembly map $\mu^*: K_{1-*}(\mathbf{c}^{\text{red}}Y) \rightarrow K^*(Y)$ is rationally surjective.

- (a) Emerson and Meyer [EM06, Corollary 8.10] proved that under these assumptions on Y its coarse co-assembly map is an isomorphism.
- (b) Engel and Wulff proved that under these assumptions on the space Y the coarse co-assembly map $\mu^*: K_{1-*}(\mathbf{c}^{\text{red}}Y) \rightarrow KX^*(Y)$ is surjective [EW17, Theorem 5.10]. Because uniform contractibility of Y implies $KX^*(Y) \cong K^*(Y)$, see Proposition 4.42, the claim follows.
- (c) Emerson and Meyer [EM06, Section 9] proved that under these assumptions on H its coarse co-assembly map $\mu^*: K_{1-*}(\mathbf{c}^{\text{red}}EH) \rightarrow KX^*(EH)$ is an isomorphism. In this case, EH is uniformly contractible and so we get $KX^*(H) \cong K^*(EH)$ by Proposition 4.42. Moreover, EH has continuously bounded geometry. \square

6.2 Groups with a γ -element

The following result is also a consequence of Theorem 6.2 and was stated as Theorem 1.7 in the introduction.

Corollary 6.5. *Let N be a finite aspherical complex, and assume that $H = \pi_1 N$ has a γ -element.*

Then for every proper metric space X endowed with a proper action of a countable discrete group G , the external product map

$$\mathcal{HR}_m^G(X) \otimes K_n(N) \rightarrow \mathcal{HR}_{m+n}^{G \times H}(X \times \tilde{N})$$

is rationally injective for each $m, n \in \mathbb{Z}$.

Proof. Since N is assumed to be a finite aspherical complex, the group H is torsion-free and \tilde{N} is an H -finite model for $EH = \underline{E}H$. Hence by Corollary 5.3 we conclude that the equivariant coarse co-assembly map

$$\mu_H^*: K_{1-*}(\mathbf{c}^{\text{red}}\tilde{N} \rtimes_{\max} H) \rightarrow K_H^*(\tilde{N}) \cong K^*(N)$$

is surjective. Thus we can apply Theorem 6.2 together with the isomorphism $K_*(N) \cong K_*^H(\tilde{N})$. \square

In the case of K-homology the above injectivity statement follows from the Künneth formula and is therefore valid in full generality, that is, without assuming the existence of a γ -element. But for the K-theory of the reduced group C*-algebras, the Künneth formula is only known to hold if $\pi_1 M$ satisfies the Baum–Connes conjecture with coefficients in any C*-algebras with trivial $\pi_1 M$ -action [CEO04], which is a considerably stronger assumption than the existence of a γ -element. We explore this in Section 7 below.

6.3 Higson-essentialness and hypereuclidean manifolds

We restate the definition of a Higson-essential manifold which was given in Definition 1.4.

Definition 6.6. We say that a complete Riemannian spin^c -manifold X of dimension m is *Higson-essential* if there exists $\vartheta \in K_{1-m}(\mathfrak{c}^{\text{red}} X)$ such that $\langle [\not{D}_X], \mu^*(\vartheta) \rangle = 1$, where $[\not{D}_X] \in K_m(X)$ denotes the K-homological fundamental class of the spin^c -structure. If the condition is relaxed to merely $\langle [\not{D}_X], \mu^*(\vartheta) \rangle \neq 0$, then X is called *rationally Higson-essential*. \blacklozenge

Start with an immediate observation.

Proposition 6.7. *Let X be a complete Riemannian spin^c -manifold X of dimension m such that the coarse co-assembly map $\mu^*: K_{1-m}(\mathfrak{c}^{\text{red}} X) \rightarrow K^m(X)$ is (rationally) surjective. Then X is (rationally) Higson-essential.*

Proof. This follows from the definition because there always exists a class $\beta_X \in K^m(X)$ such that $\langle [\not{D}_X], \beta_X \rangle = 1$. Indeed, β_X can be taken to be the Bott generator inside a coordinate patch $\mathbb{R}^m \subset X$. \square

Remark 6.8. In particular, a complete spin^c -manifold which is also a space of a type considered in Corollary 6.4 is Higson-essential. \blacklozenge

Higson-essentialness is an analytic condition. We will contrast it with the following coarse geometric property which is a generalization of hypereuclidean manifolds from Definition 1.2.

Definition 6.9 (compare [BH10, Definition 2.11]). We say a complete oriented Riemannian manifold X of dimension m is called *coarsely hypereuclidean* if there exists a coarse map $\varphi: X \rightarrow \mathbb{R}^m$ such that $\varphi_*(c[X]) \in \text{HX}_m(\mathbb{R}^m)$ is a generator, where HX denotes coarse homology and $c: H_*^{\text{lf}}(X) \rightarrow \text{HX}_*(X)$ is the coarsification map. If the condition is relaxed to merely $\varphi_*(c[X]) \neq 0 \in \text{HX}_m(\mathbb{R}^m)$, then X is called *rationally coarsely hypereuclidean*.

Moreover, we say that X is (rationally) *stably coarsely hypereuclidean* if there is $k \in \mathbb{N}$ such that $X \times \mathbb{R}^k$ is (rationally) coarsely hypereuclidean. \blacklozenge

Since any proper Lipschitz map is coarse and the coarsification map is an isomorphism for \mathbb{R}^m , it is immediate that (rationally, stably) hypereuclidean in the sense of Definition 1.2 implies (rationally, stably) coarsely hypereuclidean.

6.3.1 Coarsely hypereuclidean implies Higson-essential

In this subsection, we prove that stably coarsely hypereuclidean spin^c -manifolds are Higson-essential. To deal with the *stable* aspect, we need to work with the suspension isomorphism for the stable Higson corona. We discuss this in the following remark.

Remark 6.10 (Suspension for the stable Higson corona). Let $\mathbb{R}_- := (-\infty, 0]$ and $\mathbb{R}_+ := [0, \infty)$. Then $X \times \mathbb{R} = X \times \mathbb{R}_- \cup X \times \mathbb{R}_+$ is a coarsely excisive cover by closed subsets. We then have three pull-back diagrams of C^* -algebras of the form

$$\begin{array}{ccc} \mathfrak{C}(X \times \mathbb{R}) & \longrightarrow & \mathfrak{C}(X \times \mathbb{R}_+) \\ \downarrow & & \downarrow \\ \mathfrak{C}(X \times \mathbb{R}_-) & \longrightarrow & \mathfrak{C}(X), \end{array}$$

where $\mathfrak{C} \in \{C_0(-, \mathcal{K}), \bar{\mathfrak{c}}^{\text{red}}, \mathfrak{c}^{\text{red}}\}$. Moreover, all maps in these diagrams are surjections. These properties follow from [Wil13, Lemmas 3.3 and 3.4]. Associated to each of these pullback diagrams we have a long exact Mayer–Vietoris sequence in K-theory [Bla98, Section 21.2]. As $X \times \mathbb{R}_{\pm}$ is flasque, $K_*(\mathfrak{C}(X \times \mathbb{R}_{\pm})) = 0$. Thus the boundary maps in the Mayer–Vietoris sequences yield isomorphisms $\Sigma: K_*(\mathfrak{C}(X)) \xrightarrow{\cong} K_{*-1}(\mathfrak{C}(X \times \mathbb{R}))$. Applying the functorial exact sequence $0 \rightarrow C_0(-, \mathcal{K}) \rightarrow \bar{\mathfrak{c}}^{\text{red}}(-) \rightarrow \mathfrak{c}^{\text{red}}(-) \rightarrow 0$ yields an exact sequence of pullback diagrams. Thus the co-assembly map, which is the boundary map associated to $0 \rightarrow C_0(-, \mathcal{K}) \rightarrow \bar{\mathfrak{c}}^{\text{red}}(-) \rightarrow \mathfrak{c}^{\text{red}}(-) \rightarrow 0$, commutes with the Mayer–Vietoris boundary map up to a sign. In other words, we have the diagram

$$\begin{array}{ccc} K^*(X) & \xrightarrow{\Sigma} & K^{*+1}(X \times \mathbb{R}) \\ \mu^* \uparrow & & \mu^* \uparrow \\ K_{1-*}(\mathfrak{c}^{\text{red}} X) & \xrightarrow{\Sigma} & K_{1-(**+1)}(\mathfrak{c}^{\text{red}}(X \times \mathbb{R})) \end{array}$$

which commutes up to multiplication with -1 . ◆

We are now ready to prove the main theorem of this subsection.

Theorem 6.11. *Let X be an m -dimensional (rationally) stably coarsely hyper-euclidean spin^c -manifold. Then X is (rationally) Higson-essential.*

Proof. By assumption, we have a coarse map $\varphi: X \times \mathbb{R}^k \rightarrow \mathbb{R}^{m+k}$ such that $\varphi_* c[X \times \mathbb{R}^k] = d c[\mathbb{R}^{m+k}]$, where $d = 1$ (or $d \neq 0 \in \mathbb{Z}$, respectively). We first observe that then the same applies to the K-homological fundamental classes, that is,

$$\varphi_* c[\not{D}_{X \times \mathbb{R}^k}] = d c[\not{D}_{\mathbb{R}^{m+k}}] \in \text{KX}_{m+k}(\mathbb{R}^{m+k}).$$

This follows from the Chern character. Indeed,

$$\text{ch}_*([\not{D}_{X \times \mathbb{R}^k}]) \in \bigoplus_{i \in \mathbb{N}} H_{m+k-2i}^{\text{lf}}(X \times \mathbb{R}^k; \mathbb{Q})$$

is the Poincaré dual of the Todd class $\text{Td}(X \times \mathbb{R}^k)$ of $X \times \mathbb{R}^k$. Since \mathbb{R}^{m+k} has coarse and locally finite homology only in degree $m+k$, only the top degree component of $\text{ch}_*([\not{D}_{X \times \mathbb{R}^k}])$ contributes to $\text{ch}_*(\varphi_* c[\not{D}_{X \times \mathbb{R}^k}])$. As the degree zero part of $\text{Td}(X \times \mathbb{R}^k)$ is 1, this component is $[X \times \mathbb{R}^k]$ and we conclude that

$$\text{ch}_*(\varphi_* c[\not{D}_{X \times \mathbb{R}^k}]) = \varphi_* c[X \times \mathbb{R}^k] = d c[\mathbb{R}^{m+k}] \in \text{HX}_{m+k}(\mathbb{R}^{m+k}; \mathbb{Q}).$$

This also implies the desired integral equality because the transformation $\mathbb{Z} \cong \text{HX}_*(\mathbb{R}^{m+k}) \rightarrow \text{HX}_*(\mathbb{R}^{m+k}; \mathbb{Q}) \cong \mathbb{Q}$ is injective.

Next, we consider the following commutative diagram (up to signs).

$$\begin{array}{ccccc}
K^m(X) & \xrightarrow[\cong]{\Sigma^k} & K^{m+k}(X \times \mathbb{R}^k) & & K^{m+k}(\mathbb{R}^{m+k}) \\
\uparrow c^* & & \uparrow c^* & & \uparrow c^* \cong \\
KX^m(X) & \xrightarrow[\cong]{\Sigma^k} & KX^{m+k}(X \times \mathbb{R}^k) & \xleftarrow{\varphi^*} & KX^{m+k}(\mathbb{R}^{m+k}) \\
\uparrow \mu^* & & \uparrow \mu^* & & \uparrow \mu^* \cong \\
K_{1-m}(\mathfrak{c}^{\text{red}} X) & \xrightarrow[\cong]{\Sigma^k} & K_{1-m-k}(\mathfrak{c}^{\text{red}}(X \times \mathbb{R}^k)) & \xleftarrow{\varphi^*} & K_{1-m-k}(\mathfrak{c}^{\text{red}} \mathbb{R}^{m+k})
\end{array}$$

Here Σ^k signifies the k -fold application of the suspension isomorphism from Remark 6.10. Let $\beta_{\mathbb{R}^{m+k}} \in K^{m+k}(\mathbb{R}^{m+k})$ be the Bott generator and $\vartheta_{\mathbb{R}^{m+k}} \in K_{1-m-k}(\mathfrak{c}^{\text{red}} \mathbb{R}^{m+k})$ such that $c^* \mu^* \vartheta_{\mathbb{R}^{m+k}} = \beta_{\mathbb{R}^{m+k}}$. Further, we set $\vartheta \in K_{1-m}(\mathfrak{c}^{\text{red}} X)$ to be the unique element such that $\Sigma^k \vartheta = \varphi^* \vartheta_{\mathbb{R}^{m+k}}$. To prove that X is (rationally) Higson-essential, we need to verify that $\langle [\mathcal{D}_X], c^* \mu^* \vartheta \rangle = \pm d$. Indeed, $\Sigma^k \mu^* \vartheta = \pm \mu^* \Sigma^k \vartheta = \pm \mu^* \varphi^* \vartheta_{\mathbb{R}^{m+k}} = \pm \varphi^* \mu^* \vartheta_{\mathbb{R}^{m+k}}$ and therefore

$$\begin{aligned}
\langle [\mathcal{D}_X], c^* \mu^* \vartheta \rangle &= \langle [\mathcal{D}_X] \times [\mathcal{D}_{\mathbb{R}^k}], \Sigma^k c^* \mu^* \vartheta \rangle \\
&= \langle [\mathcal{D}_X] \times [\mathcal{D}_{\mathbb{R}^k}], c^* \Sigma^k \mu^* \vartheta \rangle \\
&= \pm \langle [\mathcal{D}_{X \times \mathbb{R}^k}], c^* \varphi^* \mu^* \vartheta_{\mathbb{R}^{m+k}} \rangle \\
&= \pm \langle \varphi_* c[\mathcal{D}_{X \times \mathbb{R}^k}], \mu^* \vartheta_{\mathbb{R}^{m+k}} \rangle \\
&= \pm \langle d c[\mathcal{D}_{\mathbb{R}^{m+k}}], \mu^* \vartheta_{\mathbb{R}^{m+k}} \rangle \\
&= \pm \langle d[\mathcal{D}_{\mathbb{R}^{m+k}}], c^* \mu^* \vartheta_{\mathbb{R}^{m+k}} \rangle \\
&= \pm \langle d[\mathcal{D}_{\mathbb{R}^{m+k}}], \beta_{\mathbb{R}^{m+k}} \rangle = \pm d. \quad \square
\end{aligned}$$

6.3.2 Injectivity of external products

The following result explains why the notion of Higson-essentialness is useful for our purposes. It was stated as Theorem 1.5 in the introduction.

Corollary 6.12. *Let Y be an n -dimensional spin^c -manifold of continuously bounded geometry. Suppose that Y is (rationally) Higson-essential (in particular, this is satisfied if Y is (rationally) stably coarsely hypereuclidean). Assume furthermore that Y is endowed with a proper action of a countable discrete group H which preserves the spin^c structure.*

Then for every proper metric space X which is endowed with a proper action of a countable discrete group G , the external product

$$\mathcal{HR}_*^G(X) \xrightarrow{\times[\mathcal{D}_Y]} \mathcal{HR}_{*+n}^{G \times H}(X \times Y)$$

is (rationally) split-injective.

Proof. If $[\mathcal{D}_Y] \in K_n^H(Y)$ denotes the equivariant fundamental class of Y , then $F_*[\mathcal{D}_Y] \in K_n(Y)$ is the fundamental class of the underlying non-equivariant spin^c -manifold, where we used the notation from Section 5.5. By assumption, there exists $\vartheta \in K_{1-n}(\mathfrak{c}^{\text{red}} Y)$ with $\langle F_*[\mathcal{D}_Y], \mu^* \vartheta \rangle = d$, where $d = 1$ (or $d \neq 0$, respectively). Let $\iota: \mathfrak{c}^{\text{red}} Y \rightarrow \mathfrak{c}^{\text{red}} Y \rtimes_{\mu} H$ be the map which was also considered in Section 5.5. Then Definition 5.25, Theorem 5.23, and Proposition 5.30 imply that

$$\langle [\mathcal{D}_Y], \mu_{H^*}^* \iota_* \vartheta \rangle = \langle F_*[\mathcal{D}_Y], \mu^* \vartheta \rangle = d.$$

Thus we can apply Theorem 6.1. □

6.3.3 Contractible manifolds

Assuming that the non-compact manifold is contractible (for instance, consider the universal cover of an aspherical manifold), we get the following stronger results.

Proposition 6.13. *Let Y be a contractible n -dimensional spin^c -manifold. Suppose that Y is (rationally) Higson-essential (in particular, this is satisfied if Y is (rationally) stably coarsely hypereuclidean).*

Then the coarse co-assembly map $\mu^: \mathbf{K}_{1-*}(\mathbf{c}^{\text{red}}Y) \rightarrow \mathbf{K}^*(Y)$ is (rationally) surjective.*

Proof. Because Y is contractible, Poincaré duality for spin^c -manifolds (see for instance [HR00b, Ex. 11.8.11]) implies

$$\mathbf{K}^p(Y) \cong \text{RK}_{n-p}(Y) \cong \text{RK}_{n-p}(*) \cong \begin{cases} \mathbb{Z} & p \equiv n \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

If Y is Higson-essential, by definition there exists a class in the image of the co-assembly map which pairs to 1 with $[\not{D}_Y]$. Such a class must generate $\mathbf{K}^n(Y) \cong \mathbb{Z}$. If Y is only rationally Higson-essential, the co-assembly map is still rationally non-trivial and hence rationally surjective as the target is one-dimensional. \square

Corollary 6.14. *Let Y be a contractible n -dimensional spin^c -manifold of continuously bounded geometry. Suppose that Y is rationally Higson-essential (in particular, this is satisfied if Y is rationally stably coarsely hypereuclidean).*

Then for every proper metric space X endowed with a proper action of a countable discrete group G , the external product

$$\mathcal{HR}_m^G(X) \otimes \mathbf{K}_p(Y) \rightarrow \mathcal{HR}_{m+p}^G(X \times Y)$$

is rationally injective for each $m, p \in \mathbb{Z}$.

Proof. Combine Proposition 6.13 and Theorem 6.2. \square

7 Künneth theorems for the structure group

In this final section—which does not use the methods of the rest of the paper—we deduce a full Künneth-like theorem for the analytic structure group in the case that the Baum–Connes conjecture is satisfied.

Theorem 7.1. *Let H be a countable discrete group. Assume that H is torsion-free and satisfies the Baum–Connes conjecture for all coefficient C^* -algebras with trivial H -action.³⁶*

Then for any simplicial complex M , the external product map

$$\text{RS}_*^G(\widetilde{M}) \otimes \text{RK}_*(BH) \rightarrow \text{RS}_*^{G \times H}(\widetilde{M} \times EH),$$

where $G = \pi_1 M$, is rationally an isomorphism. If $\text{RK}_(BH)$ is torsion-free, then it is integrally an isomorphism.*

³⁶For example, H could be a-T-menable [HK01] or it could be hyperbolic [Laf12; Pus12].

Proof. We consider the following commutative diagram which is part of a map between rationally exact sequences.

$$\begin{array}{ccccc}
\mathrm{RS}_*^G(\widetilde{M}) \otimes \mathrm{RK}_*(BH) & \longrightarrow & \mathrm{RK}_*(M) \otimes \mathrm{RK}_*(BH) & \longrightarrow & \mathrm{K}_*(\mathrm{C}_{\mathrm{red}}^*G) \otimes \mathrm{RK}_*(BH) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{RS}_*^{G \times H}(\widetilde{M} \times EH) & \longrightarrow & \mathrm{RK}_*(M \times BH) & \longrightarrow & \mathrm{K}_*(\mathrm{C}_{\mathrm{red}}^*(G \times H))
\end{array}$$

Indeed, the top sequence is the analytic exact sequence (1.7) for the space \widetilde{M} tensored with $\mathrm{RK}_*(BH)$. The functor $- \otimes \mathrm{RK}_*(BH)$ is rationally exact and hence the top sequence is rationally an exact sequence. The lower sequence is the analytic exact sequence (1.7) for the space $\widetilde{M} \times EH$.

The middle vertical arrow is rationally an isomorphism due to the Künneth formula for K-homology. Note that in general for any ring spectrum we have a Künneth spectral sequence. But in the case of the complex K-theory spectrum one can show that the spectral sequence degenerates suitably to give rise to a short exact sequence. This follows by similar arguments as presented in the remark on top of page 62 in [Lan16].

Because we assume H to be torsion-free and to satisfy the Baum–Connes conjecture, the assembly map $\mathrm{RK}_*(BH) \rightarrow \mathrm{K}_*(\mathrm{C}_{\mathrm{red}}^*H)$ is an isomorphism. Also, the reduced group C*-algebra $\mathrm{C}_{\mathrm{red}}^*H$ satisfies the Künneth formula, because we assume that H satisfies (the reduced version of) the Baum–Connes conjecture for all coefficient C*-algebras with trivial H -action.³⁷ Hence the right vertical arrow in the diagram is rationally an isomorphism.

It follows from the five lemma that the left vertical arrow must also be rationally an isomorphism.

If $\mathrm{RK}_*(BH)$ is torsion-free, then the functor $- \otimes \mathrm{RK}_*(BH)$ is integrally exact, that is, the top sequence in the above diagram is exact. Furthermore, the Tor-terms in the Künneth formulas for the middle and right vertical arrows vanish in this case and therefore these arrows are isomorphisms. Therefore the left vertical arrow is also an isomorphism by the five-lemma. \square

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³⁷Tu [Tu99] proved that if H is amenable, then $\mathrm{C}_{\mathrm{red}}^*H$ lies in the bootstrap class and hence satisfies the Künneth formula. That $\mathrm{C}_{\mathrm{red}}^*H$ satisfies the Künneth formula if H satisfies the Baum–Connes conjecture was proven by Chabert, Echterhoff, and Oyono-Oyono [CEO04].

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